

# RANDOM FIELDS AND THE GEOMETRY OF WIENER SPACE\*

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## Abstract

In this work, we consider infinite dimensional extensions of some finite dimensional Gaussian geometric functionals called the Gaussian Minkowski functionals. These functionals appear as coefficients in the probability content of a tube around a convex set  $D \subset \mathbb{R}^k$  under the standard Gaussian law  $N(0, I_{k \times k})$ . Using these infinite dimensional extensions, we consider geometric properties of some smooth random fields in the spirit of [2] that can be expressed in terms of reasonably smooth Wiener functionals.

## 1 Introduction and Motivation

We start with a description of a certain class of set functionals determined by the canonical Gaussian measure on  $\mathbb{R}^k$ . By canonical, we shall mean centered and having covariance  $I_{k \times k}$ . Its density with respect to Lebesgue measure on  $\mathbb{R}^k$  is therefore given by  $(2\pi)^{-k/2} e^{-\|x\|^2/2}$ .

The set functionals, formally defined in (1.5), arise in certain statistical problems involving a smoothly parameterized set of regression models. Our motivating example begins therefore with a reasonably smooth subset  $M \subset \mathbb{R}^n$  which smoothly indexes a family of linear regression models. Such linear models are often used in statistical analysis of fMRI data as championed by the late Keith Worsley [25, 27, 26]. More precisely, for each  $x \in M$ , consider a linear regression model given by

$$Y_i(x) = \sum_{j=1}^p a_{ij} \beta_j(x) + Z_i(x), \quad i = 1, \dots, n, \quad (1.1)$$

where, for each  $x \in M$ ,  $Y_i(x)$  is an observation corresponding to the  $i$ -th subject,  $(\beta_j(x))$  is a  $p$ -vector of unknown coefficients and  $(a_{ij})$  is a design matrix

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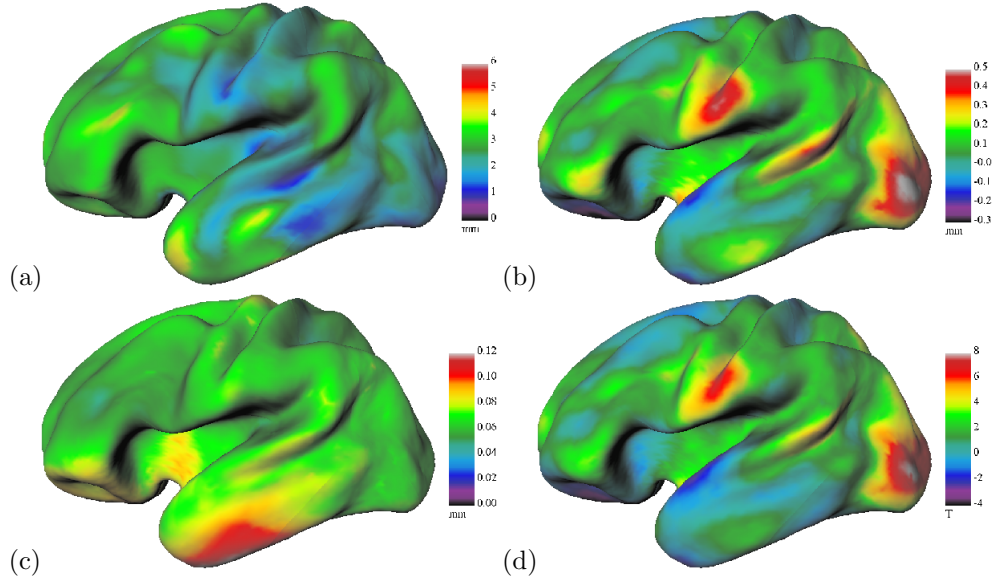


Figure 1.1: Cortical thickness (mm) of  $n = 321$  normal adults on the average cortical surface. (a) Smoothed cortical thickness of a single subject; (b) average cortical thickness of the 163 males minus the 158 females; (c) pooled within-gender standard deviation of the difference in (b) ( $df=319$ ); (d) two-sample T-statistic, (b) divided by (c), ( $df=319$ ).

(known), and finally  $(Z_1(\cdot), \dots, Z_n(\cdot))$  are independent copies of a mean zero, unit variance, Gaussian random field. Above,  $M$  plays the role of the surface of an *average/standard* brain, and  $(Y_i(x))$  are some measurements corresponding to various subjects, at each point  $x$  on the surface of the brain. The function  $\beta : M \rightarrow \mathbb{R}^p$  represents coefficients of various parameters in the regression model (1.1). For instance,  $(Y_1(x), \dots, Y_n(x))$  could be the cortical thickness,  $(a_1, \dots, a_n)$  could be a particular covariate such as the IQ score or gender. In this case,  $\beta(x) \in \mathbb{R}$  represents the coefficient of linear dependence of the covariate of interest and the cortical thickness at some location  $x \in M$ . A more realistic model would also control for covariates with perhaps  $\beta_1(x)$  being the first coordinate of  $\beta(x)$  in such a model. Figure 1 from [19] depicts a typical application of such models. In this figure, the covariate of interest is chosen to be gender as a demonstrative example. The reader is referred to the works of [19, 25, 27, 26] for more detailed examples of applications.

Once the model is set, a common approach in such models is to consider a family of hypothesis tests  $(H_{0,x})_{x \in M}$  indexed by  $M$ . This is typically done in the usual regression fashion, by testing various contrasts, or affine functions of  $(\beta_j(x))$ . For instance, the null hypothesis  $H_{0,x} : \beta_{1,x} = 0$  is typically tested via

the  $T$ -statistic

$$T(x) = \frac{\widehat{\beta}_{1,x}}{SE(\widehat{\beta}_{1,x})}$$

with  $SE(\widehat{\beta}_{1,x})$  being the usual unbiased estimate of the standard deviation of  $\widehat{\beta}_{1,x}$ . Under  $H_{0,x}$ , standard results show that  $T(x)$  has a Student's  $T$  distribution with  $n - p$  degrees of freedom. Under the intersection null  $H_0 = \cap_{x \in M} H_{0,x}$ , the random field marginally has a  $T$  distribution at each location and has a particular structure. Such a random field was termed a  $T$  random field in [25] with  $n - p$  degrees of freedom. Following the discussion in [19], one way of approaching the multiple testing problem with hypotheses  $(H_{0,x})_{x \in M}$  is to apply Roy's Union-Intersection Principle and use maximum of the random field to calibrate the Family Wise Error Rate. In particular, finding  $t_\alpha$  such that

$$P_{H_0} \left( \sup_{x \in M} T(x) \geq t_\alpha \right) \approx \alpha$$

should be powerful at detecting small regions with marked departure from  $H_{0,x}$  within the region. Above,  $H_0$  indicates that the probability should be computed under the intersection null  $H_0 = \cap_{x \in M} H_{0,x}$ . In what follows, all calculations will be carried out under the intersection null so we will drop the reference to  $H_0$ . In fact, in what follows  $H$  will usually refer to the Cameron-Martin space of an abstract Wiener space and will have nothing to do with hypothesis testing after the end of this section.

## 1.1 Expected Euler characteristic heuristic

Using the Euler characteristic heuristic developed by Robert Adler and Keith Worsley (c.f. e.g. [1, 25, 27, 26]) and described in great length [2], one can approximate the above probability by  $E(\chi(A_t(T; M)))$ , where  $A_t(T; M) = \{x \in M : T(x) \geq t\} \subset M$ , and  $\chi$  is the Euler-Poincaré characteristic, which brings us to the central theme of this paper.

Let  $M$  be an  $m$ -dimensional reasonably smooth manifold, with  $(\xi_1, \dots, \xi_k)$  identically and independently distributed copies of Gaussian random field defined on  $M$ . Subsequently, for any  $F : \mathbb{R}^k \rightarrow \mathbb{R}$ , with two continuous derivatives, we can define a new random field on  $M$  given by  $f(x) = F(\xi_1(x), \dots, \xi_k(x))$ , for each  $x \in M$ . For instance, the test statistic  $T(x)$  in the cortical thickness example above can be expressed as some fixed function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  of the  $(Z_1(x), \dots, Z_n(x))$  under  $H_{0,x}$  with the same function being applied at each  $x \in M$ . Hence, the entire random field  $(T(x))_{x \in M}$  can be expressed in the form  $T(x) = F(Z_{1,x}, \dots, Z_{n,x})$ .

Using the above Euler characteristic heuristic for approximating  $P$ -value for appropriately large values of  $u$ , and Theorem 15.9.5 of [2], we have

$$P \left( \max_{x \in M} f(x) \geq u \right) \approx E(\chi(A_u(f; M))) = \sum_{j=0}^m (2\pi)^{-j/2} \mathcal{L}_j(M) \mathcal{M}_j^{\gamma_k}(F^{-1}[u, \infty)), \quad (1.2)$$

where  $\mathcal{L}_j(M)$ , for  $j = 0, 1, \dots, m$  are the Lipschitz-Killing curvatures (LKC) of the manifold  $M$  defined with respect to the metric *induced*<sup>1</sup> by  $\xi_1$  on  $M$ , and  $\mathcal{M}_j^{\gamma^k}(F^{-1}[u, \infty))$  for  $j = 0, 1, \dots$  are the Gaussian Minkowski functionals (GMFs) of the set  $F^{-1}[u, \infty) \subset \mathbb{R}^k$ .

The LKCs for a large class<sup>2</sup> of subsets of any finite dimensional Euclidean space, can be defined via a *Euclidean tube formula*. In particular, let  $M \subset \mathbb{R}^k$  be an  $m$ -dimensional set with convex support cone, then writing  $\lambda_k$  as the standard  $k$ -dimensional Euclidean measure,  $B_k$  as the  $k$ -dimensional unit ball centered at origin, for small enough values of  $\rho$ , we have

$$\lambda_k(M + \rho B_k) = \sum_{j=0}^m \frac{\pi^{(n-j)/2}}{\Gamma(\frac{(n-j)}{2} + 1)} \rho^{n-j} \mathcal{L}_j(M) = \sum_{j=0}^m \frac{\rho^{n-j}}{(n-j)!} \theta_{n-j}(M), \quad (1.3)$$

where  $\mathcal{L}_j(M)$  is the  $j$ -th LKC of the set  $M$  with respect to the usual Euclidean metric, and  $\theta_j(M)$ 's are called the Minkowski functionals of the set  $M$ . Note the relation between LKCs and the Minkowski functionals. At first glance, the simple change in the ordering, connecting the LKCs and the Minkowski functionals, appears redundant, but a closer look at the two sets of functionals reveals that the LKCs are invariant of the space into which  $M$  is embedded [24], whereas Minkowski functionals are not.

Geometrically, LKCs for a smooth  $(k-1)$ -dimensional manifold  $M$  embedded in  $\mathbb{R}^k$ , can be defined via integrals of various functions of the eigenvalues of the corresponding shape operator, which are also called *principal curvatures*. In particular,  $\mathcal{L}_{k-1}(M)$  is the  $(k-1)$ -dimensional Lebesgue measure of the set  $M$ , and the other LKCs can be defined as

$$\mathcal{L}_j(M) = \frac{1}{s_{k-j}(k-1-j)!} \int_{\partial M} P_{k-1-j}(\lambda_1(x), \dots, \lambda_{k-1}(x)) \mathcal{H}_{k-1}(dx),$$

where  $s_j$  is the surface area of a unit ball in  $\mathbb{R}^j$  ( $\lambda_1(x), \dots, \lambda_{k-1}(x)$ ) are the principal curvatures at  $x \in \partial M$ , obtained using the outward unit normal field, and  $P_i(\lambda_1(x), \dots, \lambda_{k-1}(x))$  is the  $i$ -th symmetric polynomial in  $(k-1)$  indices. In case, when the set  $M$  is not unit codimensional, then the definition involves another integral over the normal bundle. Using the relationship in (1.3), we can similarly define the Minkowski functionals using the principal curvatures. These LKCs and Minkowski functionals also give rise to what are commonly known as *generalized curvature measures*, defined on the sphere bundle  $S(M)$  which we can identify with a subset of  $S(\mathbb{R}^k) \times \mathbb{R}^k$  which we can take to be the sphere bundle of  $\mathbb{R}^k$  with the Euclidean metric. Formally, the generalized curvature measures induced by the LKCs of a smooth manifold  $M$  of unit codimension,

<sup>1</sup>Let  $X$  and  $Y$  be two vector fields on  $M$ , then the metric  $g$  induced by a real valued Gaussian random field  $\xi$  is defined as  $g(X, Y) = E(X\xi Y\xi)$ , where  $X\xi$  and  $Y\xi$  are the directional derivatives of  $\xi$ .

<sup>2</sup>The tube formula holds true for all sets of *positive reach*, as defined in [4], which are essentially sets with convex support cones (see [2]).

are given as

$$\begin{aligned} \mathcal{L}_j(M; A \times B) \\ = \frac{1}{s_{k-j}(k-1-j)!} \int_{\partial M \cap B} 1_A(-\eta(x)) P_{k-1-j}(\lambda_1(x), \dots, \lambda_{k-1}(x)) \mathcal{H}_{k-1}(dx), \end{aligned} \quad (1.4)$$

where  $A \times B \subset S(\mathbb{R}^k) \times \mathbb{R}^k$  is any Borel set and  $\eta$  is the outward pointing unit normal vector of  $\partial M$ . When the set  $\partial M$  has codimension higher than 1, the integral over the normal bundle induces another measure on  $S^{k-1}$ . In the above example, this integral over the sphere in the normal space simply evaluates to  $1_A(-\eta(x))$ . The corresponding Minkowski generalized curvature measures are given as  $\Theta_j(M; A) \triangleq (j! \omega_j) \mathcal{L}_{k-j}(M; A)$ , where  $\omega_j$  is the volume of a unit ball in  $\mathbb{R}^j$ .

The generalized curvature measures defined this way, are therefore signed measures induced by the Lebesgue measure of the ambient space. By replacing Lebesgue measure in (1.3), by an appropriate Gaussian measure, we can define a parallel Gaussian theory. In particular, let us again start with a smooth set  $M \subset \mathbb{R}^k$ , together with the canonical Gaussian measure  $\gamma_k$  on  $\mathbb{R}^k$ . For this measure, we could consider computing the probability content of a tube around  $M$ , leading us to a *Gaussian tube formula* which we state as

$$\gamma_k(M + \rho B_k) = \gamma_k(M) + \sum_{j=1}^{\infty} \frac{\rho^j}{j!} \mathcal{M}_j^{\gamma_k}(M), \quad (1.5)$$

where  $\mathcal{M}_j^{\gamma_k}(M)$  is the  $j$ -th GMF of the set  $M$ . If  $M$  is compact and convex, i.e. if  $M$  is a convex body, then we can take the right hand side (1.5) to be a power series expansion for the left hand side. For certain  $M$ , this expansion must be taken to be a formal expansion, in the sense that up to terms of some order, the left and right hand side above agree. For example, if  $M$  is a centrally-symmetric cone such as the rejection region for a  $T$  or  $F$  statistic, then  $M$  has a singularity at the origin in the sense that the geometric structure of the cone around 0 is non-convex and the expansion above is accurate only up to terms of size  $O(\rho^{n-1})$ .

These GMFs can also be expressed as integrals with respect to *generalized curvature measures* induced by the Minkowski functionals defined above. In particular,

$$\mathcal{M}_j^{\gamma_k}(M) \triangleq (2\pi)^{-k/2} \sum_{m=0}^{j-1} \binom{j-1}{m} \Theta_{m+1}(M, H_{j-1-m}(\langle \eta, x \rangle) e^{-|x|^2/2}), \quad (1.6)$$

where  $\Theta_{m+1}(M, H_{j-1-m}(\langle \eta, x \rangle) e^{-|x|^2/2})$  is the integral of  $H_{j-1-m}(\langle \eta, x \rangle) e^{-|x|^2/2}$  with respect to the  $(m+1)$ -th generalized Minkowski curvature measure, and  $H_k(y)$  is the  $k$ -th Hermite polynomial in  $y$ .

We refer the reader to [2] for more rigorous geometric definitions of Minkowski functionals, generalized curvature measures and so on.

## 1.2 Our object of study: a richer class of random fields

In this paper we intend to extend (1.2) to a larger class of random fields  $f$ , which can be expressed using  $F : C_0[0, 1] \rightarrow \mathbb{R}$ , where  $C_0[0, 1]$  is the space of continuous functions  $f : [0, 1] \rightarrow \mathbb{R}$ , such that  $f(0) = 0$ , also referred to as the *classical Wiener space*, when equipped with the standard Wiener measure on this sample space. In other words, we shall consider random fields which can be expressed as some smooth *Wiener functional*. For instance, let us start with a smooth manifold  $M$  together with a Gaussian field  $\{B^x(t) : t \in \mathbb{R}_+, x \in M\}$  defined on it, such that the covariance function is given by

$$E(B^x(t)B^y(s)) = s \wedge t C(x, y), \quad (1.7)$$

such that  $C : M \times M \rightarrow \mathbb{R}$  is assumed to be a smooth<sup>3</sup> function. This infinite dimensional random field can be used to construct many more random fields on  $M$ . For instance:

**Example 1.1 (Stochastic integrals)** *Let  $V : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function, and consider the following random field*

$$f(x) = \int_0^1 V(B^x(s)) dB^x(s) = F(B^x(\cdot)) \quad (1.8)$$

where  $F : C_0 \rightarrow \mathbb{R}$  is the Wiener functional

$$F(\omega) = \left( \int_0^1 V(B(t)) dB(t) \right) (\omega).$$

This is clearly an extension of the random fields in (1.2). As a consequence of our extension of the Gaussian Minkowski functionals to smooth Wiener functionals, we prove that, under suitable smoothness conditions on  $V$

$$\mathbb{E}(\chi(A_u(f; M))) = \sum_{j=0}^{\dim(M)} (2\pi)^{-j/2} \mathcal{L}_j(M) \mathcal{M}_j^\mu(F^{-1}[u, +\infty)).$$

Our smoothness conditions are rather strong in this paper: we assume  $V$  is  $C^4$  with essentially polynomial growth. We need such strict assumptions to ensure regularity of various conditional densities derived from the random field (1.8) and its first two derivatives at a point  $x \in M$ .

A quick look at (1.2) reveals that in order to extend it to the case when  $F : C_0[0, 1] \rightarrow \mathbb{R}$ , we must be able to define GMFs for infinite dimensional subsets of  $C_0[0, 1]$ , as  $F^{-1}[u, \infty) \subset C_0[0, 1]$ . In the present form, i.e. (1.6), the definition of GMFs appears to depend on the summability of the principal curvatures of the set  $\partial(F^{-1}[u, \infty))$  at each point  $x \in \partial(F^{-1}[u, \infty))$  as well as the

<sup>3</sup>More precise statements regarding the order of smoothness of  $C$  will appear later in Section 6, where we actually prove an extension of (1.2)

integrability of these sums. In infinite dimensions this summability requirement is equivalent to an operator being trace class. This is quite a strong requirement, and may be very hard to check. Indeed, the natural summability requirements of operators in the natural infinite dimensional calculus on  $C_0$ , the Malliavin calculus, is Hilbert-Schmidt class rather than trace class.

Therefore, we shall first modify the definition of GMFs, from (1.6) to one which is more amenable for an extension to the infinite dimensional case. This will be done in Section 2.

After setting up the notations and some technical background on the Wiener space in Section 3, the all important step, that of extending the appropriate definition of GMFs to the case of codimension one *convex*<sup>4</sup>, smooth subsets of the Wiener space, is accomplished in Section 5. The characterization of GMFs in infinite dimensional case, will be done precisely the same way as in the case of finite dimensions, where, as noted earlier, the GMFs are identified as the coefficients appearing in the Gaussian tube formula.

Finally, in Section 6, we use the infinite dimensional extension of the GMFs to obtain an extension of (1.2), for random fields which can be expressed as stochastic integrals driven by  $B^x(\cdot)$  as defined in Example 1.1, and discuss other possible implications of the extension.

**Remark 1.2** *Most of our methods are invariant to the formulation of the random field as a stochastic integral. Hence, should a random field satisfy all the regularity conditions appearing in Section 6, we expect our methods to work smoothly, though with a few changes.*

## 2 Preliminaries I: the finite dimensional theory

In this section we shall use the standard finite dimensional theory of transformation of measure for Gaussian spaces to establish the Gaussian tube formula, and thereby identify the GMFs. In the process, we shall modify the definition (1.6) of the GMFs to one which is more suited to extension to the infinite dimensional case.

We begin by recalling some well-known facts about finite dimensional Malliavin calculus, emphasizing that standard objects, such as the Malliavin divergence, are really just Riemannian objects. We shall start with  $\mathbb{R}^k$  equipped with the Riemannian metric

$$\langle \cdot, \cdot \rangle_{\gamma^k} = \exp(-\|x\|^2/k) \langle \cdot, \cdot \rangle,$$

where  $\langle \cdot, \cdot \rangle$  denotes the standard Euclidean Riemannian metric. It is easy to see that the Riemannian measure induced by this metric is the Gaussian measure (after appropriate normalization). Let  $V(x) = \sum_{i=1}^k V_i(x) E_i(x)$  be a vector field on  $\mathbb{R}^k$ , where  $(E_i(x))$  is the canonical orthonormal basis of the tangent

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<sup>4</sup>More precisely we should say  $H$ -convex here, as our sets will be the excursion sets of  $H$ -convex Wiener functionals [5]

space at  $x \in \mathbb{R}^k$ . The Riemannian divergence of  $V$ , under this metric, evaluated at point  $x \in \mathbb{R}^k$  is given by

$$(\operatorname{div}(V))(x) = \sum_{i=1}^k \frac{\partial V_i(x)}{\partial x_i} - \sum_{i=1}^k V_i(x)x_i.$$

For reasons, made explicit in the subsequent sections, we shall denote<sup>5</sup>  $\delta(V) = (-1)\operatorname{div}(V)$ .

Following is the key theorem that connects the definition (1.6) of GMFs to its refinement, which we shall use in extension to the infinite dimensional case.

**Theorem 2.1** *Let  $\gamma_k$  be the Gaussian measure on  $\mathbb{R}^k$  and  $T$  be a mapping from  $\mathbb{R}^k$  into itself, given by  $T = I_{\mathbb{R}^k} + u$ , where  $I_{\mathbb{R}^k}$  is the identity map on  $\mathbb{R}^k$ , and  $u : \mathbb{R}^k \rightarrow \mathbb{R}^k$  is Sobolev differentiable and  $|u(x) - u(y)| \leq c(\rho)|x - y|$  for any  $x, y \in \mathbb{R}^k$  with  $|x - y| < \rho$ . Then, the Radon–Nikodym derivative of  $\gamma_k \circ T$  with respect to the measure  $\gamma_k$  is given by*

$$\frac{d\gamma_k \circ T}{d\gamma_k} = |\det_2(I_{\mathbb{R}^k} + \nabla u)| \exp\left(-\delta(u) - \frac{1}{2}\|u\|^2\right),$$

where  $\|\cdot\|$  is the usual Euclidean norm, and  $\det_2$  is the generalized Carleman–Fredholm determinant given by

$$\det_2(I_{\mathbb{R}^k} + \nabla u) = \prod_{i=1}^k (1 + \lambda_i) e^{-\lambda_i},$$

where  $\{\lambda_i\}_{i=1}^k$  are the Eigen values of the operator  $\nabla u$ .

This, and a much stronger result with Sobolev differentiability of  $u$  replaced by the local Sobolev differentiability, can be proven using the standard theory of transformation of Gaussian measure on  $\mathbb{R}^k$ , which can be found in Chapters 3 and 4 of [22].

Subsequently, for a smooth, unit codimensional, convex set  $A \subset \mathbb{R}^k$ , let us define the tube  $\operatorname{Tube}(A, \rho)$ , of width  $\rho$  around the set  $A$  as the set  $(A \oplus B(0, \rho))$ , where  $B(0, \rho)$  is the  $k$ -dimensional ball of radius  $\rho$  centred at origin, and  $\oplus$  is used to denote the *Minkowski sum* of sets. Next, we shall define a signed distance function given by

$$d_{\partial A}(x) = \begin{cases} \inf_{y \in \partial A} \|y - x\| & \text{for } x \notin A \\ 0 & \text{for } x \in \partial A \\ -\inf_{y \in \partial A} \|y - x\| & \text{for } x \in \operatorname{Int}(A) \end{cases}$$

where  $\operatorname{Int}(A)$  denotes the interior of the set  $A$ .

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<sup>5</sup>Note that  $\delta(V)$  is synonymous to the finite dimensional version of the Malliavin divergence. Therefore,  $\delta(E_i)$ , for  $1 \leq i \leq k$ , are i.i.d., unit variance, Gaussian random variables with mean 0.



Applying the co-area formula

$$\begin{aligned}\gamma_k(\text{Tube}(A, \rho)) &= \gamma_k(A) + \int_0^\rho \int_{d_A^{-1}(r)} \|\nabla d_{\partial A}\|^{-1/2} \frac{\exp(-\|x\|^2/2)}{(2\pi)^{n/2}} dx dr \\ &= \gamma_k(A) + \int_0^\rho \int_{d_A^{-1}(r)} \frac{\exp(-\|x\|^2/2)}{(2\pi)^{n/2}} dx dr.\end{aligned}\tag{2.9}$$

For  $r < \rho$  fixed, we can now apply Theorem 2.1 with any suitable transformation  $T_r : \mathbb{R}^k \rightarrow \mathbb{R}^k$  that agrees with

$$x \mapsto x + r \eta_x$$

on  $\{y : d_A(y) \in (-\nu, \rho)\}$  for some small positive  $\nu$ . Any such transformation maps  $\text{Tube}(d_A^{-1}(r), \epsilon)$  to  $\text{Tube}(\partial A, \epsilon)$  for  $r < \rho$  and any  $\epsilon < \nu$ . Two further applications of the co-area formula yield

$$\begin{aligned}\int_{d_A^{-1}(r)} \frac{\exp(-\|x\|^2/2)}{(2\pi)^{n/2}} dx &= \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_{\text{Tube}(d_A^{-1}(r), \epsilon)} \frac{\exp(-\|y\|^2/2)}{(2\pi)^{n/2}} dy \\ &= \lim_{\epsilon \downarrow 0} \left( \frac{1}{2\epsilon} \int_{\text{Tube}(\partial A, \epsilon)} \det_2(I_{\mathbb{R}^k} + r \nabla^2 d_{\partial A}) \right. \\ &\quad \left. \exp\left(-r\delta(\nabla d_{\partial A}) - \frac{1}{2}r^2\right) \frac{\exp(-\|y\|^2/2)}{(2\pi)^{n/2}} dy \right) \\ &= \int_{\partial A} \det_2(I_{\mathbb{R}^k} + r \nabla^2 d_{\partial A}) \\ &\quad \exp\left(-r\delta(\nabla d_{\partial A}) - \frac{1}{2}r^2\right) \frac{\exp(-\|x\|^2/2)}{(2\pi)^{n/2}} dx.\end{aligned}$$

Therefore, equation (2.9) simplifies to

$$\begin{aligned}\gamma_k(\text{Tube}(A, \rho)) &= \gamma_k(A) + \int_0^\rho \int_{\partial A} \det_2(I_{\mathbb{R}^k} + r \nabla^2 d_{\partial A}) \exp\left(-r\delta(\nabla d_{\partial A}) - \frac{1}{2}r^2\right) \\ &\quad \times \frac{\exp(-\|x\|^2/2)}{(2\pi)^{n/2}} dx dr.\end{aligned}\tag{2.10}$$

Using a yet-to-be justified Taylor series expansion of the integrand appearing in the above integral, we can finally rewrite the GMFs as

$$\begin{aligned}\mathcal{M}_{j+1}^{\gamma_k}(A) &= \int_{\partial A} \frac{d^j}{d\rho^j} \left( \det_2(I + \rho \nabla \eta) \exp(-\rho \delta(\eta) - \rho^2/2) \right) \Big|_{\rho=0} \\ &\quad \times da^{\partial A}(x),\end{aligned}\tag{2.11}$$

where  $\eta = \nabla d_{\partial A}$ , is the outward unit normal vector field to the set  $\partial A$  at point  $x \in \partial A$ ,  $\nabla \eta$  is the gradient of the vector field  $\eta$ , and  $da^{\partial A}$  is the surface measure of the set  $\partial A$ .

**Remark 2.2** *Note that in the above expression, we have removed the modulus around the  $\det_2$  part, which can be justified by taking reasonably small values of  $\rho$ .*

Clearly, for the existence of GMFs, as it appears in (2.11), it suffices to have  $\sum_i \lambda_i^2 < \infty$ , which is a much weaker condition than the summability of the  $\lambda_i$ 's, and as will be seen in the subsequent sections a more natural condition in the Malliavin calculus. This makes (2.11) stand out as the appropriate candidate for the extension to the infinite dimensional case<sup>6</sup>.

We shall note here that, exactly the same approach, together with a few changes, can be applied to define GMFs of piecewise smooth manifolds, which shall be presented with complete details in Sections 4 and 5.

### 3 Preliminaries II: the infinite dimensional theory

In this section, we recall some established concepts in Malliavin calculus which we shall need in later sections. We begin with an abstract Wiener space  $(X, H, \mu)$ , where  $H$ , equipped with the inner product  $\langle \cdot, \cdot \rangle_H$ , is a separable Hilbert space, called the Cameron–Martin space,  $X$  is a Banach space into which  $H$  is injected continuously and densely, and finally  $\mu$  is the standard cylindrical Gaussian measure on  $H$ . For the sake of simplicity, one can appeal to the classical case when we have  $H$  as the space of real valued, absolutely continuous functions on  $[0, 1]$  with  $L^2([0, 1])$  derivatives, which is continuously embedded in  $X = C_0([0, 1])$  the space of real valued continuous functions  $f$  on  $[0, 1]$ , such that  $f(0) = 0$ .

#### 3.1 Sobolev spaces on Wiener space

Let us denote by  $\mathcal{S}(E)$  the space of smooth  $E$  valued random variables, for some Hilbert space  $E$ , such that a random variable  $F \in \mathcal{S}(E)$  has the form

$$F = f(\langle h_1, \omega \rangle, \dots, \langle h_n, \omega \rangle),$$

where  $h_1, \dots, h_n \in H$ ,  $f \in C_b^\infty(\mathbb{R}^n; E)$ , and  $n = 1, 2, \dots$ . Typically, we shall take  $E$  as  $\mathbb{R}^k$  or  $\otimes^m H$ . In particular, for  $F \in \mathcal{S}(\mathbb{R})$ , the Gross–Sobolev derivative of a real valued  $F$  is defined as an  $H$ -valued random variable given by

$$DF = \sum_{i=1}^n \partial_i f(\langle h_1, \omega \rangle, \dots, \langle h_n, \omega \rangle) h_i.$$

Similarly, one can also define the Gross–Sobolev derivative of an  $\otimes^m H$  valued  $F$  as an  $\otimes^{m+1} H$  valued random variable, written as  $DF$ . The derivative defined

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<sup>6</sup>For a detailed, historical and technical, account on the issue of choosing the second definition as the appropriate one to extend to the infinite dimensional case, we refer the reader to the introduction of Chapter 3 of [22].

above is sometimes also referred to as Shigekawa's  $H$ -derivative. Considering it as an operator from  $\mathcal{S}(\mathbb{R})$  to  $\mathcal{S}(H)$ , it can be seen (cf. [9]) that the operator is closable from  $L^p(X)$  to  $L^p(X; H)$  for any  $p \geq 1$ .

Writing  $D^k F = D(D \dots (D(F)))$ , we shall define  $D_k^p(X; \mathbb{R})$  as the space of functions  $f \in L^p(X)$  such that

$$\|f\|_{D_k^p} = \|f\|_{L^p(X)} + \sum_{j=1}^k \|D^j f\|_{L^p(X; \otimes^j H)} < \infty, \quad (3.12)$$

for  $p > 1$  and  $k = 1, 2, \dots$ . This definition can naturally be extended to  $D_k^p(X; \mathbb{R}^d)$  with the understanding that for  $E$ , a Hilbert space equipped with the norm  $\|\cdot\|_E$ , the appropriate norm for  $g \in L^p(X; E)$  is defined as

$$\|g\|_{L^p(X; E)} = (E(\|g\|_E^p))^{1/p}.$$

Writing  $\delta$  for the dual of  $D$  under the Gaussian measure, the Ornstein–Uhlenbeck operator  $L$  can be written as,  $L = -\delta D$ . Using the well known Krée–Meyer inequalities, we can equivalently define the space  $D_k^p(X; \mathbb{R})$  as the class of functions  $f \in L^p(X)$  such that

$$\|f\|_{p,k} = \|(I - L)^{k/2} f\|_{L^p(X)} < \infty.$$

This second definition lets us extend the definition of the Sobolev spaces  $D_\alpha^p(X; \mathbb{R})$ , to fractional, as well as negative, values of  $\alpha$  (cf. [23]). The Sobolev spaces  $D_{-\alpha}^p(X; \mathbb{R})$  for  $\alpha > 0$ , as in the case of Euclidean space, are the spaces of distributions, defined as the dual of  $D_\alpha^q(X; \mathbb{R})$  where, as usual,  $\frac{1}{q} + \frac{1}{p} = 1$ . Throughout this paper, whenever appropriate, we will adopt this convention.

The space of test Wiener functionals defined above as  $\mathcal{S}(\mathbb{R})$ , is equivalent to

$$D_\infty^{\infty-} \triangleq \bigcap_{\alpha > 0} \bigcap_{1 < p < \infty} D_\alpha^p,$$

and its dual, the space of *generalized Wiener functionals*, is given by

$$D_{-\infty}^{1+} \triangleq \bigcup_{\alpha > 0} \bigcup_{1 < p < \infty} D_{-\alpha}^p.$$

Consequently, let us define the analogous *infinitely integrable* random variables as

$$L^{\infty-}(X; \mathbb{R}) \triangleq \bigcap_{1 < p < +\infty} L^p(X; \mathbb{R}).$$

Furthermore, we shall write  $D_\alpha^{\infty-}(X; \mathbb{R}) = \bigcap_{1 < p < \infty} D_\alpha^p(X; \mathbb{R})$ , and  $D_{-\alpha}^{1+}(X; \mathbb{R}) = \bigcup_{1 < p < \infty} D_{-\alpha}^p(X; \mathbb{R})$ .

Finally, we shall end this section with another definition which translates to the regularity of Wiener functionals.

**Definition 3.1** *For an  $\mathbb{R}^k$  valued Wiener functional  $F = (F_1, \dots, F_k)$ , the Malliavin covariance (matrix)  $\sigma^F = (\langle DF_i, DF_j \rangle_H)_{ij}$ , and the functional  $F$  itself, is called nondegenerate in the sense of Malliavin if  $(\det \sigma^F)^{-1} \in L^{\infty-}$ , whenever  $\det \sigma^F$  is well defined.*

### 3.2 H-Convexity

As is shown in [2], the GMFs are well defined for finite dimensional Whitney stratified manifolds. In order to characterize the class of subsets of the Wiener space, for which we shall define the GMFs, we shall recall the notion of  $H$ -convexity.

**Definition 3.2** *An  $H$ -convex functional is defined as a measurable functional  $F : X \rightarrow \mathbb{R} \cup \{\infty\}$  such that for any  $h, k \in H$ ,  $\alpha \in [0, 1]$*

$$F(\omega + \alpha h + (1 - \alpha)k) \leq \alpha F(\omega + h) + (1 - \alpha)F(\omega + k) \quad a.s. \quad (3.13)$$

We shall note here that,  $H$ -convex functionals encompass a fairly large class of Wiener functionals. For instance, let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a real, convex function, then  $F(\omega) = \int_0^1 f(\omega_t) dt$ , is a valid  $H$ -convex Wiener functional, whenever the integral is well defined.

Some of the properties of  $H$ -convex functionals are listed below.

1. Any measurable convex functional defined on  $X$ , is  $H$ -convex.
2. To verify  $H$ -convexity, it suffices to verify the condition (3.13) for  $k = -h$  and  $\alpha = 1/2$ .
3. If  $\{F_n\}_{n \in \mathbb{N}}$  is a sequence of  $H$ -convex functionals converging in probability, then the limit is also  $H$ -convex.
4.  $F \in L^p$  for some  $p > 1$  is  $H$ -convex if and only if  $D^2 F$  is a positive and symmetric Hilbert–Schmidt operator valued distribution on  $X$ .

For the proofs of the above properties, we refer the reader to [20, 21, 22]. We shall discuss more about  $H$ -convex functionals in the later sections.

### 3.3 Quasi-sure analysis

In this section, we shall resolve some technical aspects of defining integrals of Wiener functionals with respect to measures concentrated on  $\mu$ -zero sets. Clearly, all Wiener functionals are *de facto* defined up to  $\mu$ -zero sets. Therefore, in order to be able to define the integral of Wiener functionals with respect to measures which are concentrated on sets where the functionals are not very well defined, we must resort to what is referred to as *quasi-sure analysis*.

The building blocks for quasi-sure analysis on the Wiener space are the *capacities*, which are defined in accordance with the Sobolev spaces  $D_\alpha^p$  defined above. Following Malliavin [8] and Takeda [17], we shall define various capacities as follows.

**Definition 3.3** *Let  $1 < p < \infty$  and  $\alpha > 0$ . For an open set  $O$  of  $X$ , we define its  $(p, \alpha)$ -capacity  $C_\alpha^p(O)$  by*

$$C_\alpha^p(O) = \inf \{ \|U\|_{p,\alpha} : U \in D_\alpha^p(X; \mathbb{R}), U \geq 1 \text{ } \mu - a.e. \text{ on } O \}.$$

For each subset of  $A$  of  $X$ , we define its  $(p, \alpha)$ -capacity  $C_\alpha^p$  by

$$C_\alpha^p(A) = \inf \{C_\alpha^p(O) : O \text{ is open and } O \supset A\}.$$

These capacities are *finer* scales to estimate the size of sets in  $X$  than  $\mu$ . In particular, a set of  $(p, \alpha)$ -capacity zero is always a  $\mu$ -zero set, but the converse is not true in general. Moreover for any open set  $O \subset X$ , and  $0 < \alpha < \beta$

$$[\mu(O)]^{1/p} = C_0^p(O) \leq C_a^p(O) \leq C_\beta^p(O),$$

which follows from the hierarchical embedding of the Sobolev spaces  $D_r^p(X; \mathbb{R})$ , as a function of  $p$  and  $r$ .

A property  $\pi$  is said to be true  $(p, \alpha)$ -quasi-everywhere (q.e.), if

$$C_\alpha^p(\pi \text{ is not satisfied}) = 0.$$

**Definition 3.4** If  $(p, \alpha)$ -capacity of a set  $A$  vanishes for all  $1 < p < \infty$  and  $\alpha > 0$ , then the set  $A$  is said to be a slim set<sup>7</sup>.

**Remark 3.5** The capacities defined above, as is clear from the definition, are also connected to the Sobolev distributions. Let us define

$$\Delta_\alpha^p = \{\nu \in D_{-\alpha}^q : \nu \text{ is a positive measure, and } \|\nu\|_{q, -\alpha} \leq 1\},$$

where  $q^{-1} + p^{-1} = 1$ . Then, for any Borel set  $A$  of the Wiener space,

$$C_\alpha^p(A) = \sup_{\nu \in \Delta_\alpha^p} \nu(A).$$

**Remark 3.6** Another way of defining capacities of sets is, by first defining capacities of Wiener functionals i.e., for a  $[0, \infty]$ -valued lower semi-continuous (l.s.c.) Wiener functional  $h$ , its  $(p, \alpha)$ -capacity is defined as

$$C_\alpha^p(h) = \inf \{\|g\|_{p, \alpha} : g \in D_\alpha^p(X; \mathbb{R}), g \geq h \text{ } \mu - \text{a.e.}\}.$$

Then, for any  $[-\infty, \infty]$ -valued Wiener functional  $f$ , its  $(p, \alpha)$ -capacity is defined as

$$C_\alpha^p(f) = \inf \{C_\alpha^p(h) : h \text{ is l.s.c. and } h(x) \geq |f(x)| \forall x\}$$

One of the most crucial steps in obtaining the *co-area* formula in the Wiener space, which in turn is a necessary step to obtain the *tube-formula* in the Wiener space, is to be able to extend ordinary Wiener functionals which are defined up to  $\mu$ -zero sets, to sets of capacity zero. Quasi-sure analysis lets us do precisely that, and much more.

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<sup>7</sup>For a detailed account of this and more on redefinitions and quasi-sure analysis, we refer the reader to Chapter IV of [8].

**Definition 3.7** A measurable functional  $F$  is said to have a  $(p, \alpha)$ -redefinition  $F^*$ , satisfying

$$F^* = F \quad \mu - \text{a.s.}, \text{ and}$$

$$F^* \text{ is } (p, \alpha)\text{-quasi-continuous,}$$

if for all  $\epsilon > 0$ , there exists an open set  $O_\epsilon$  of  $X$ , such that

$$C_\alpha^p(O_\epsilon) < \epsilon$$

and the restriction of  $F^*$  to the complement set  $O_\epsilon^c$  is continuous under the norm of uniform convergence on  $X$ .

It can easily be seen that two redefinitions of the same functional, differ only on a set of  $(p, \alpha)$ -capacity zero, thereby implying the uniqueness of a  $(p, \alpha)$ -redefinition upto  $(p, \alpha)$ -capacity zero sets.

Connecting the redefinition to the order of integrability and differentiability of the functional, is the following theorem, the proof of which can be found in [8].

**Theorem 3.8** Every functional  $F \in D_\alpha^p(X; \mathbb{R}^k)$  has a  $(p, \alpha)$ -quasi-continuous redefinition, which can be taken to be in the first Baire class.

**Remark 3.9** A  $(p, \alpha)$ -redefinition of  $F$  is sometimes also referred to as the  $(p, \alpha)$ -quasi-continuous version of  $F$ .

In what follows in the remainder of this section, we recall some facts from the Malliavin calculus that will be helpful in our description of a tube below.

If  $\alpha > 1$ , one can make a statement similar to Theorem 3.8 related to the differentiability of  $F \in D_\alpha^p(X; \mathbb{R})$ , essentially a form of Taylor's theorem with remainder.

**Lemma 3.10** Suppose  $F \in D_\alpha^p(X; \mathbb{R})$ ,  $\alpha > 1$ . Then, for each  $h \in H$

$$\frac{1}{\epsilon} (F(x + \epsilon h) - F(x)) - \langle DF(x), h \rangle_H \xrightarrow{D_{\alpha-1}^{p_1}(X; \mathbb{R})} 0.$$

for any  $p_1 < p$ .

**Proof:** Define

$$\begin{aligned} X_{n,h} &= n (F(x + h/n) - F(x)) \in D_{\alpha-1}^{p_1} \\ &= \Lambda^{\alpha-1} Y_{n,h}, \quad Y_{n,h} \in L^{p_1} \end{aligned}$$

where  $\Lambda = (I - L)^{-1/2}$  is the inverse of the Cauchy operator [8]. For each  $h \in H$ ,  $X_{n,h}$  converges in  $L^{p_1}$ , so the Kree-Meyer inequalities imply that  $Y_{n,h}$  also converges in  $L^{p_1}$ . A second application of the Kree-Meyer inequalities imply that

$$\|Y_{n,h} - Y_{m,h}\|_{L^p} \approx \|X_{n,h} - X_{m,h}\|_{D_{\alpha-1}^{p_1}}.$$

Or,  $X_{n,h}$  is Cauchy in  $D_{\alpha-1}^{p_1}$  hence its limit  $\langle DF(x), h \rangle_H \in D_{\alpha-1}^{p_1}$ .  $\square$

Hence, by the Borel-Cantelli property for the capacities  $C_\alpha^{p_1}$  (Corollary IV.1.2.4 of [8]) for each  $h \in H$  we can extract a sequence  $\varepsilon_n(h)$  such that

$$C_{\alpha-1}^{p_1} \left( \left\{ x : \lim_{n \rightarrow \infty} \frac{1}{\varepsilon_n(h)} (F(x + \varepsilon_n(h)h) - F(x)) = \langle DF(x), h \rangle \right\}^c \right) = 0. \quad (3.14)$$

**Corollary 3.11** *Suppose  $F \in D_\alpha^p(X; \mathbb{R})$ ,  $\alpha > 1$  is non-degenerate and  $H_\infty \subset H$  is a countable dense subset. Then,*

$$C_{\alpha-1}^{p_1} \left( \left\{ x : DF(x) \neq 0, \forall h \in H_\infty \exists \varepsilon_n(h) \rightarrow 0 \text{ such that} \right. \right. \\ \left. \left. \lim_{n \rightarrow \infty} \left( \frac{1}{\varepsilon_n(h)} (F^*(x + \varepsilon_n(h)h) - F^*(x)) - \langle DF^*(x), h \rangle_H \right) = 0 \right\}^c \right) = 0. \quad (3.15)$$

**Proof:** The only thing that needs verifying beyond what was pointed out above is that

$$C_{\alpha-1}^{p_1} (\{x : DF(x) \neq 0\}^c) = 0.$$

This follows from the Tchebycheff inequality (Theorem IV.2.2 of [8]) applied to  $\|DF\|_H \in D_{\alpha-1}^{p_1}$  and a Borell-Cantelli argument.  $\square$

There is an obvious higher order version of the Taylor's theorem above which we will use the second order version in our description of the tube below. If we are willing to sacrifice some moments, we can further specify in Corollary 3.11 that the existence of the partial derivatives of  $F$  as a limit at  $x$  implies their existence as limits at  $x + h$  for all  $h \in H_\infty$ ,  $\|h\| \leq K$  for some fixed, large  $K$ .

**Corollary 3.12** *Suppose  $F \in D_\alpha^p(X; \mathbb{R})$  and  $H_\infty \subset H$  is a countable dense subset. Then, for all  $p_1 < p$*

$$C_{\alpha-1}^{p_1} \left( \left\{ x : \forall h_1, h_2 \in H_\infty, \|h_1\| \leq K \exists \varepsilon_n(h_1, h_2) \rightarrow 0 \text{ such that} \right. \right. \\ \left. \left. \lim_{n \rightarrow \infty} \left( \frac{1}{\varepsilon_n(h_1, h_2)} (F^*(x + h_1 + \varepsilon_n(h_1, h_2)h_2) - F^*(x + h_1)) - \langle DF^*(x + h_1), h_2 \rangle_H \right) = 0 \right\}^c \right) = 0. \quad (3.16)$$

**Proof:** This follows from the fact that the translation operator

$$f(\cdot) \xrightarrow{T_h} f(\cdot + h)$$

is a continuous map from  $D_\alpha^p$  to  $D_\alpha^{p_1}$  for any  $p_1 < p$  which follows directly from the Cameron-Martin theorem.  $\square$

**Remark 3.13** *Finally, we note that we can, by choosing  $H_\infty$  appropriately, choose the set, say,  $A$  in Corollary 3.12 in such a way that  $y \in A$  and  $y + \varepsilon h \in A$  for all  $h \in H_\infty$  and for all  $\varepsilon$  in some countable dense subset of  $\mathbb{R}$ .*

## 4 Key ingredients for a tube formula

In this section we shall adopt a step-wise approach to reach our first goal, that of obtaining a (Gaussian) volume of tube formula, for reasonably smooth subsets of the Wiener space. The three main steps are: (i) characterizing subsets of the Wiener space via Wiener functionals, for which tubes, and thus GMFs, are well defined; (ii) assurance that the surface measures are well defined for the sets defined via the Wiener functionals; and finally, (iii) a change of measure formula for surface area measures corresponding to the lower dimensional surfaces of the Wiener space.

We shall first characterize the functionals for which the surfaces measures are well defined, subsequently we shall prove a change of measure formula for the surfaces defined via such functionals. Finally, we shall define the class of sets for which the tube formula and GMFs are well defined by imposing more regularity conditions on the Wiener functionals.

### 4.1 The Wiener surface measures

Before we start discussing the infinite dimensional version of a *surface measure*, it is indeed a good exercise to look back to the finite dimensional theory, and see what one means by a surface measure with respect to an underlying measure. As is evident from equation (2.11), the Gaussian surface measure of an  $(n - 1)$  dimensional hypersurface  $\mathcal{I}$  in  $\mathbb{R}^n$  is given by

$$\int_{\mathcal{I}} f da^{\mathcal{I}} = \int_{\mathcal{I}} f(x) \frac{\exp(-\|x\|^2/2)}{(2\pi)^{n/2}} \mathcal{H}_{n-1}(dx),$$

Let us start with a reasonably smooth,  $\mathbb{R}^k$  valued Wiener functional  $F = (F_1, \dots, F_k)$ . For  $\mathbf{u} = (u_1, \dots, u_k) \in \mathbb{R}^k$ , we write  $Z_{\mathbf{u}} = \cap_{i=1}^k F_i^{-1}(u_i)$ . The sets  $\{Z_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{R}^k}$  define a foliation of hypersurfaces imbedded in  $X$ .

The surface measures of these foliations  $Z_{\mathbf{u}}$  are closely related to the density  $p_F$  of the push-forward measure  $F_*(\mu)$  on  $\mathbb{R}^k$  with respect to the Lebesgue measure on  $\mathbb{R}^k$ . On  $\mathbb{R}^n$  this comes directly from the co-area formula: given  $F : \mathbb{R}^n \rightarrow \mathbb{R}^k$  Lipschitz

$$\int_{\mathbb{R}^n} \|F(y)\| g(y) \frac{\exp(-\|y\|^2/2)}{(2\pi)^{n/2}} dy = \int_{\mathbb{R}^k} \int_{F^{-1}(\mathbf{z})} g(x) \frac{\exp(-\|x\|^2/2)}{(2\pi)^{n/2}} \mathcal{H}_{n-k}(dx) \mathcal{H}_k(d\mathbf{z})$$

where

$$\|F(y)\| = \sqrt{\det(\nabla F_i, \nabla F_j)_{1 \leq i, j \leq k}}$$

is the Jacobean of the map  $F : \mathbb{R}^n \rightarrow \mathbb{R}^k$ . Inserting a Dirac delta on both sides



yields

$$\begin{aligned}
& \int_{\mathbb{R}^n} \delta_{\mathbf{u}}(F(y)) \|F(y)\| G(y) \frac{\exp(-\|y\|^2/2)}{(2\pi)^{n/2}} dy \\
&= \int_{\mathbb{R}^k} \delta_{\mathbf{u}}(\mathbf{z}) \int_{F^{-1}(\mathbf{z})} G(x) \frac{\exp(-\|x\|^2/2)}{(2\pi)^{n/2}} \mathcal{H}_{n-k}(dx) \mathcal{H}_k(d\mathbf{z}) \\
&= \int_{F^{-1}(\mathbf{u})} G(x) \frac{\exp(-\|x\|^2/2)}{(2\pi)^{n/2}} \mathcal{H}_{n-k}(dx).
\end{aligned}$$

That is, we can recover the integral over a particular level set  $F^{-1}(\mathbf{u})$  using distributions or generalized functions, the result of which is to identify the submanifold  $F^{-1}(\mathbf{u})$  with  $\delta_{\mathbf{u}} \circ F$ . This approach, which we now describe, can be extended to infinite dimensions using the notion of generalized Wiener functionals.

Heuristically, the density  $p_F$  can be defined as

$$p_F(\mathbf{u}) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^k \omega_k} \mu(F^{-1}B(\mathbf{u}, \epsilon)) = \frac{1}{\epsilon^k \omega_k} \int_X 1_{B(\mathbf{u}, \epsilon)}(F(x)) \mu(dx),$$

where  $B(\mathbf{u}, \epsilon)$  is an  $\epsilon$ -radius ball in  $\mathbb{R}^k$ , centered at  $\mathbf{u}$  with volume  $\epsilon^k \omega_k$ . Noting that

$$1_{B(\mathbf{u}, \epsilon)}(\cdot) / (\epsilon^k \omega_k) \approx \delta_{\mathbf{u}}(\cdot) \quad (4.17)$$

the Dirac delta function at  $u \in \mathbb{R}^k$ , leads to an alternative definition of the density as

$$p_F(\mathbf{u}) = E(\delta_{\mathbf{u}} \circ F), \quad (4.18)$$

as long as we can make sense of the composition  $\delta_{\mathbf{u}} \circ F$ . For a smooth, real valued Wiener functional  $G$ , we also expect the following relation to hold

$$E[G \delta_{\mathbf{u}} \circ F] = E^{F=\mathbf{u}}(G) \times p_F(\mathbf{u}), \quad (4.19)$$

where  $E^{F=\mathbf{u}}(G)$  is the conditional expectation of  $G$  given  $F = \mathbf{u}$ , assuming the composition  $\delta_{\mathbf{u}} \circ F$  is well-defined.

Making this heuristic calculation rigorous leads us back to the Sobolev spaces of Section 3.1 where the object  $\delta_{\mathbf{u}} \circ F$  is related to a *generalized Wiener functional*, that is, an element of some  $D_{-\alpha}^p$  for  $p > 1, \alpha > 0$  through the pairing

$$\langle G, \delta_{\mathbf{u}} \circ F \rangle_{D_{\alpha}^q, D_{-\alpha}^p} = E[G \delta_{\mathbf{u}} \circ F]$$

representing conditional expectation given  $F = \mathbf{u}$  for any  $G \in D_{\alpha}^q$ . What is left to determine is, for a given  $F$  which Sobolev spaces contain  $\delta_{\mathbf{u}} \circ F$ .

The following theorem, the proof of which can be found in [23], provides the answer, taking us one step closer to defining the surface measure corresponding to the conditional expectation.

**Theorem 4.1** *Let  $F$  be an  $\mathbb{R}^k$  valued, nondegenerate Wiener functional such that  $F \in D_{1+\epsilon}^{\infty-}(X; \mathbb{R}^k)$  for  $\epsilon > 0$ , and the density  $p_F$  of the law of  $F$  is bounded. Also, let  $0 \leq \beta < \min(\epsilon, \alpha)$  and  $1 < p < \infty$  satisfy*

$$1 < p < \frac{k}{\max\{(k + \beta - \min(\alpha, \epsilon)), 0\}}, \quad (4.20)$$

*and finally,  $\mathcal{O} = \{z \in \mathbb{R}^k : p_F(z) > 0\}$ . Then for  $G \in D_\alpha^q(X; \mathbb{R})$ , with  $\frac{1}{p} + \frac{1}{q} = 1$ , we have*

$$\zeta(u) = E(G \delta_u \circ F) \in W_\beta^q(\mathcal{O}), \quad (4.21)$$

*where  $W_\beta^q(\mathcal{O})$  is the Sobolev space of real valued, weak  $\beta$ -differentiable functions which are  $q$ -integrable.*

Recall that for  $F = (F_1, \dots, F_k) \in D_{1+\epsilon}^{\infty-}$ , the density  $p_F \in W_\epsilon^{\infty-}(\mathcal{O})$  (cf.[9]). Now using the differentiability of the density  $p_F$  together with equations (4.19), (4.21), and the algebraic structure of the Sobolev spaces we have

$$E^F(G) \in W_\beta^q(\mathcal{O}),$$

for any  $G \in D_\alpha^q(X; \mathbb{R})$  where the function  $E^F(G)$  is defined as

$$E^F(G)(\mathbf{u}) = E^{F=\mathbf{u}}(G).$$

That is, for each  $F \in D_{1+\epsilon}^{\infty-}(X; \mathbb{R})$  there exists a continuous mapping

$$E^F : D_\alpha^q(X; \mathbb{R}) \rightarrow W_\beta^q(\mathcal{O}).$$

This in turn induces a dual map

$$(E^F)^* : W_{-\beta}^p(\mathcal{O}) \rightarrow D_{-\alpha}^p(X; \mathbb{R}) \quad (4.22)$$

defined via the dual relationship

$$\langle E^F(G), v \rangle_{W_\beta^q(\mathcal{O}), W_{-\beta}^p(\mathcal{O})} = \langle G, (E^F)^* v \rangle_{D_\alpha^q(X; \mathbb{R}), D_{-\alpha}^p(X; \mathbb{R})}. \quad (4.23)$$

Informally, this map, sometimes referred to as the Watanabe map (see Section 6 of Chapter III of [8]), is just composition, i.e.

$$(E^F)^* v = v \circ F.$$

The object  $(E^F)^* \delta_{\mathbf{u}}$  is almost the surface measure needed in (2.11) but it is just a generalized Wiener functional, i.e. distribution on  $X$ , at this point. If we are to justify our Taylor series expansion via a dominated convergence argument, we need to know that it has a representation as a measure on  $X$ .

Clearly, for positive  $G \in D_\alpha^q(X; \mathbb{R})$ , we shall have

$$\langle G, (E^F)^* \delta_{\mathbf{u}} \rangle_{D_\alpha^q(X; \mathbb{R}), D_{-\alpha}^p(X; \mathbb{R})} = E^{F=\mathbf{u}}(G) > 0.$$

Therefore,  $(E^F)^*\delta_{\mathbf{u}} \in D_{-\alpha}^p(X; \mathbb{R})$  defines a *positive generalized Wiener functional* which in turn, by using Theorem 4.3 of Sugita [16], implies that there exists a finite positive Borel measure  $\nu^{F, \mathbf{u}}$  on  $X$ , such that

$$E^{F=\mathbf{u}}(G) = \int_X G^*(x) \nu^{F, \mathbf{u}}(dx),$$

for all  $G \in D_{\alpha}^q(X; \mathbb{R})$ , with  $G^*$  its  $(q, \alpha)$ -quasi continuous version as defined in Section 3.3.

This proves the following corollary.

**Corollary 4.2** *Let  $F$  be such that it satisfies all the conditions of Theorem 4.1, then for each  $\mathbf{u} \in \mathcal{O}$ , there exists a probability measure  $\nu^{F, \mathbf{u}}$  defined on the Borel subsets of the Wiener space  $X$ , supported on  $F^{-1}(\mathbf{u})$ , such that, for all  $G \in D_{\alpha}^q(X; \mathbb{R})$ , we have*

$$E^{F=\mathbf{u}}(G) = \int_X G^*(x) \nu^{F, \mathbf{u}}(dx). \quad (4.24)$$

The measure  $\nu^{F, \mathbf{u}}$  defined is a probability measure on the set  $F^{-1}(\mathbf{u})$ . Using Airault and Malliavin's arguments in [3], an appropriate area measure  $da^{Z_{\mathbf{u}}}$ , corresponding to the measure  $\nu^{F, \mathbf{u}}$  can be defined as

$$\int_X G^*(x) da^{Z_{\mathbf{u}}}(x) \triangleq p_F(\mathbf{u}) \int_X G^*(x) (\det(\sigma_F))^{1/2} \nu^{F, \mathbf{u}}(dx), \quad (4.25)$$

where  $\sigma_F$  is the Malliavin covariance matrix. Note that, we attached  $Z_{\mathbf{u}}$  to the surface measure, whereas the pair  $(F, \mathbf{u})$  to the conditional probability measure. This is to emphasize that the surface measure depends only on the geometry of the set  $Z_{\mathbf{u}}$ , whereas the conditional probability measure depends on the functional from which the set is derived.

We are now in a position to justify at least part of (2.11).

**Theorem 4.3** *Let  $F$  be a  $\mathbb{R}$ -valued nondegenerate Wiener functional such that  $F \in D_{2+\epsilon}^{\infty-}(X; \mathbb{R}^k)$  and the density  $p_F$  of the law of  $F$  is bounded. Define the unit normal vector field  $\eta = DF/\|DF\|_H$ . Furthermore, suppose that*

- $E(\exp(\rho \delta(\eta))) < \infty$  for  $\rho$  in some neighbourhood of 0;
- $E(\exp(\rho^2 \|D\eta\|_{\otimes^2 H}^2)) < \infty$  for  $\rho$  in some neighbourhood of 0.

Then, for  $0 \leq \rho < \rho_c$  for some non-zero critical radius

$$\begin{aligned} & \int_0^{\rho} \int_{F^{-1}(\mathbf{u})} \det_2(I_H + r D\eta) \exp\left(-r \delta(\eta) - \frac{1}{2} r^2\right) da^{Z_{\mathbf{u}}} dr \\ &= \sum_{j \geq 1} \frac{\rho^j}{j!} \int_{F^{-1}(\mathbf{u})} \frac{d^{j-1}}{dr^{j-1}} \left( \det_2(I_H + r D\eta) \exp(-r \delta(\eta) - r^2/2) \right) \Big|_{r=0} da^{Z_{\mathbf{u}}}. \end{aligned} \quad (4.26)$$

Before proving the above theorem, we shall state a few results concerning the regularity of functions of smooth Wiener functionals.

**Proposition 4.4** *Let  $\alpha > 0$  and  $U \in D_{\alpha}^{\infty-}(X; \mathbb{R})$ .*

- *If  $\exp(U) \in L^p(X; \mathbb{R})$ , then  $\exp(U) \in D_{\alpha'}^{p'}$  where  $p' = p^2(\alpha - \alpha')$  for  $\alpha' < \alpha$ .*
- *If  $U > 0$   $\mu$  almost surely and  $1/U \in L^p$ , then  $1/U \in D_{\alpha'}^{p'}$  where  $p' = p^2(\alpha - \alpha')$  for  $\alpha' < \alpha$ .*

We shall skip the proofs of the above as these can be proved by replicating the proofs of Theorems 1.4 and 1.5 of Watanabe ([23]).

**Proof of Theorem 4.3:** This is just dominated convergence combined with the non-degeneracy of  $F$  as well as the following bound (c.f. Theorem 9.2 of [15])

$$|\det_2(I + A)| \leq \exp(C\|A\|_{\otimes^2 H}^2)$$

for some fixed  $C > 0$ .

Note that while using the dominated convergence, we are inherently assuming the *well definedness* of integrals of  $\exp(\rho \delta(\eta))$  and  $\exp(\rho^2 \|D\eta\|_{\otimes^2 H}^2)$  with respect to the surface measure  $da^{Z_u}$ , which requires

$$\exp(\rho \delta(\eta)) \in D_{\alpha}^q(X; \mathbb{R}) \text{ such that } q > 1/(\min\{\alpha, 1 + \epsilon\}) \quad (4.27)$$

$$\exp(\rho^2 \|D\eta\|_{\otimes^2 H}^2) \in D_{\alpha}^q(X; \mathbb{R}) \text{ such that } q > 1/(\min\{\alpha, 1 + \epsilon\}) \quad (4.28)$$

Now, using Theorem 1.5 of [23], we have  $\eta \in D_{1+\epsilon'}^{\infty-}$  for all  $\epsilon' < \epsilon$ . Subsequently using the above proposition together with the assumption involving the existence of exponential moments, we have  $\exp(\rho \delta(\eta)), \exp(\rho^2 \|D\eta\|_{\otimes^2 H}^2) \in D_{\epsilon''}^p(X; \mathbb{R})$ , such that  $p = (\frac{\rho\epsilon}{\rho})^2(\epsilon' - \epsilon'')$ , where  $\epsilon'' < \epsilon'$ . In order to satisfy (4.27) and (4.28), we must choose  $\epsilon'$  and  $\epsilon''$  such that  $\rho < \rho_c^2 \epsilon''(\epsilon' - \epsilon'')$ .  $\square$

**Remark 4.5** *Note that Theorem 4.3 does not say that the Gaussian measure of the tube is given by the power series in (4.26). Rather, it gives conditions on the sets  $Z_{\mathbf{u}} = F^{-1}(\mathbf{u})$  for which the coefficients in the power series are well-defined. Using these conditions allows us to define GMFs for level sets of functions that are not necessarily  $H$ -convex. However, for such functions we will lose the interpretation of the power series in (4.26) as an expansion for the Gaussian measure of the tube. This is similar to the distinction between the formal and exact versions of Weyl / Steiner tube formulae [18].*

## 4.2 Change of measure formula: a Ramer type formula for surface measures

After assuring ourselves of the existence of the surface Wiener measures, we shall now move onto proving a change of measure formula for the surface measures given by the equation (4.25).

As before, we shall start with a Wiener functional  $F \in D_{1+\epsilon}^{\infty-}(X; \mathbb{R}^k)$ <sup>8</sup>, so that we can define the surface measure using Theorem 4.1. Moreover, as in the previous subsection, we shall also assume that the set  $Z_u = \cap_{i=1}^k F_i^{-1}(u_i)$  is a smooth  $k$ -codimensional subset of the Wiener space  $X$ .

In order to obtain a change of measure formula for the lower dimensional subspaces of the Wiener space, we shall start with the standard change of measure formula on the Wiener space  $X$ .

Let us define a mapping  $T_\eta : X \rightarrow X$  given by  $T_\eta(x) = x + \eta_x$ , for some smooth  $\eta : X \rightarrow H$ . Moreover, let  $U$  be an open subset of  $X$ , and

1.  $T_\eta$  is a homomorphism of  $U$  onto an open subset of  $X$ ,
2.  $\eta$  is an  $H$  valued  $C^1$  map, and it's  $H$  derivative at each  $x \in U$  is a Hilbert–Schmidt operator.

This transformation induces two types of changes on the initial measure  $\mu$  defined on  $X$ . These two induced measures can be expressed as

$$\begin{aligned} P(A) &= \mu(T_\eta^{-1}(A)) = T_\eta^* \mu(A) \\ Q(A) &= \mu(T_\eta(A)) = (T_\eta^{-1})^* \mu(A) \end{aligned}$$

for  $A$  a Borel set of  $X$ .

Ramer's formula for change of measure on  $X$ , induced by a transformation defined on  $X$  and satisfying the above conditions, gives an expression for the Radon–Nikodym derivative of  $\mu \circ T_\eta$  with respect to  $\mu$ , and can be stated as follows

$$\begin{aligned} \frac{dQ}{d\mu} &= \frac{d(\mu \circ T_\eta)}{d\mu}(x) \\ &= |\det_2(I_H + \nabla \eta(x))| \exp(-\delta(\eta) - \frac{1}{2} \|\eta(x)\|_H^2) \\ &= Y_\eta(x), \end{aligned}$$

where,  $\delta(\eta)$  denotes the Malliavin divergence of an  $H$ -valued vector field  $\eta$  in  $X$ . The proof of this result can be found in [11]<sup>9</sup>. It is to be noted here that, for appropriately smooth transformations, a similar result for  $\frac{d\mu \circ T_\eta^{-1}}{d\mu}$  can be obtained by using the relationship between  $\frac{d\mu \circ T_\eta}{d\mu}$  and  $\frac{d\mu \circ T_\eta^{-1}}{d\mu}$  given by

$$\frac{d\mu \circ T_\eta}{d\mu}(x) = \left( \frac{d\mu \circ T_\eta^{-1}}{d\mu}(T_\eta x) \right)^{-1}.$$

<sup>8</sup> We shall impose further regularity on the functional  $F$  as and when required.

<sup>9</sup>For the original paper we refer the reader to [11], and for a detailed study of general transformation of measure on Wiener space, we refer the reader to [22].

Next, we shall use equation (4.2) to obtain a similar formula for measures concentrated on the lower dimensional subsets of the Wiener space, as defined in the previous section. But before that we first need to reformulate the surface measure as defined in (4.25).

**Theorem 4.6** *Let  $F \in D_{1+\epsilon}^{\infty-}(X; \mathbb{R}^k)$  satisfies the conditions from Theorem 4.1, and  $\alpha, \beta$  and  $p$  be as given in (4.20). Then, there exists a sequence probability measures  $\{\nu_n^{F, \mathbf{u}}\}_{n \geq 1}$  defined on Borel subsets of  $X$  such that*

- *the measures  $\{\nu_n^{F, \mathbf{u}}\}_{n \geq 1}$  are absolutely continuous with respect to the Wiener measure  $\mu$ , and*
- *the sequence  $\{\nu_n^{F, \mathbf{u}}\}_{n \geq 1}$  converges weakly to  $\nu^{F, \mathbf{u}}$ .*

**Proof:** Let us choose a sequence of positive distributions on  $\mathcal{O}$  given by  $\{v_n\}_{n \geq 1} \subset W_{-\beta}^p(\mathcal{O})$ , such that it converges to  $\delta_{\mathbf{u}}$  weakly in  $W_{-\beta}^p(\mathcal{O})$ , and that  $\int v_n(\xi) d\xi = 1$  for all  $n \geq 1$ . Then define the measures  $\nu_n^{F, \mathbf{u}}$  as

$$\int G(x) \nu_n^{F, \mathbf{u}}(dx) = \frac{1}{p_F(\mathbf{u})} \int G(x) v_n(F(x)) \mu(dx),$$

for all measurable  $G$  on  $(X, \mu)$ . In view of (4.23) we can clearly identify the restriction of the measures  $\nu_n^{F, \mathbf{u}}$  to  $D_{\alpha}^q(X; \mathbb{R})$  with  $(E^F)^* v_n$ . Now, by construction,  $\{(E^F)^* v_n\}_{n \geq 1}$  converges to  $(E^F)^* \delta_{\mathbf{u}}$  in  $D_{-\alpha}^p(X; \mathbb{R})$  and their limit is a non-negative generalized Wiener functional. Therefore, using Lemma 4.1 of [16], we see that the measures  $\nu_n^{F, \mathbf{u}}$  converge weakly to  $\nu^{F, \mathbf{u}}$ .  $\square$

Therefore, the surface (probability) measure of  $Z_{\mathbf{u}}$ , or the conditional probability measure corresponding to  $\{F = \mathbf{u}\}$ , for any  $\mathbf{u} \in \mathcal{O}$  can also be defined as

$$\int G^*(x) \nu^{F, \mathbf{u}}(dx) = \lim_n \int G^*(x) \nu_n^{F, \mathbf{u}}(dx) = \lim_n \frac{1}{p_F(\mathbf{u})} \int G^*(x) v_n(F(x)) \mu(dx), \quad (4.29)$$

for appropriate class of Wiener functionals  $G$ , which, as is noted in Corollary 4.2, depends on the regularity of  $F$ .

Let us now define a mapping  $T_{\rho, \eta} : X \rightarrow X$  given by  $T_{\rho, \eta}(x) = x + \rho \eta_x$ , for some  $\eta \in D_{1+\epsilon}^{\infty-}(X; H)$ . We shall study the change that the mapping  $T_{\rho, \eta}$  induces on the surface measure of  $F^{-1}(\mathbf{u})$ . In particular, we shall obtain a Girsanov type formula for the change of measure induced by the transformation. Note that

$$T_{\rho, \eta}(F^{-1}(\mathbf{u})) = \{x + \rho \eta_x : x \in F^{-1}(\mathbf{u})\}$$

Set  $F_{\rho, \eta} = F \circ T_{\rho, \eta}^{-1} \in D_{1+\epsilon}^{\infty-}$ , so that

$$F_{\rho, \eta}^{-1}(\mathbf{u}) = T_{\rho, \eta}(F^{-1}(\mathbf{u})) \triangleq Z_{\mathbf{u}}^{\eta, \rho}.$$

The corresponding area measure for  $Z_u^{\eta,\rho}$  can be identified for suitable  $G$  as

$$\begin{aligned}
& \int_{Z_u^{\eta,\rho}} G^* da^{Z_u^{\eta,\rho}} \\
&= p_{F_{\rho,\eta}}(\mathbf{u}) \int_{Z_u^{\eta,\rho}} G^*(y) [\det \sigma_{F_{\rho,\eta}}(y)]^{1/2} \nu^{F_{\rho,\eta},\mathbf{u}}(dy) \\
&= \lim_n p_{F_{\rho,\eta}}(\mathbf{u}) \int_X G^*(y) [\det \sigma_{F_{\rho,\eta}}(y)]^{1/2} \nu_n^{F_{\rho,\eta},\mathbf{u}}(dy) \\
&= \lim_n \int_X G^*(y) [\det \sigma_{F_{\rho,\eta}}(y)]^{1/2} v_n(F_{\rho,\eta}(y)) \mu(dy)
\end{aligned}$$

Now using the transformation  $y = T_{\rho,\eta}(x)$ , and replacing the function  $G^*(\cdot)$  by  $G^*(T_{\rho,\eta}^{-1}(\cdot))$ , and finally using the standard Ramer's formula from equation (4.2) we get

$$\begin{aligned}
& \int_{Z_u^{\eta,\rho}} G^*(T_{\rho,\eta}^{-1}y) da^{Z_u^{\eta,\rho}} \\
&= \lim_n \int_X G^*(x) [\det \sigma_{F_{\rho,\eta}}(T_{\rho,\eta}x)]^{1/2} v_n(F_{\rho,\eta}(T_{\rho,\eta}x)) Y_\rho^\eta(x) \mu(dx) \quad (4.30) \\
&= \lim_n \int_X G^*(x) [\det \sigma_{F_{\rho,\eta}}(T_{\rho,\eta}x)]^{1/2} v_n(F(x)) Y_\rho^\eta(x) \mu(dx),
\end{aligned}$$

where,  $Y^{\rho,\eta}(x)$  is the Radon–Nikodym derivative of the measure  $\mu \circ T_{\rho,\eta}$  with respect to the measure  $\mu$ , and as a result of (4.2), can be expressed as

$$Y_\rho^\eta(x) = |\det_2(I_H + \rho \nabla \eta(x))| \exp(-\rho \delta(\eta) - \frac{1}{2} \rho^2 \|\eta\|_H^2). \quad (4.31)$$

Using the definitions of  $F_\rho$ , the surface (probability and area) measures, and finally, rearranging the terms we can rewrite (4.30) as

$$\begin{aligned}
& \int_{Z_u^{\eta,\rho}} G^*(T_{\rho,\eta}^{-1}y) da^{Z_u^{\eta,\rho}} \\
&= \lim_n \int_X G^*(x) [\det \sigma_{F_{\rho,\eta}}(T_{\rho,\eta}x)]^{1/2} Y_\rho^\eta(x) v_n(F(x)) \mu(dx) \\
&= \lim_n p_F(u) \int_X G^*(x) \left( \frac{\det \sigma_{F_\rho}(T_{\rho,\eta}x)}{\det \sigma_F(x)} \right)^{1/2} Y_\rho^\eta(x) [\det \sigma_F(x)]^{1/2} \nu_n^{F,\mathbf{u}}(dx) \\
&= p_F(u) \int_{Z_u} G^*(x) \left( \frac{\det \sigma_{F_{\rho,\eta}}(T_{\rho,\eta}x)}{\det \sigma_F(x)} \right)^{1/2} Y_\rho^\eta(x) [\det \sigma_F(x)]^{1/2} \nu^{F,\mathbf{u}}(dx) \\
&= \int_{Z_u} G^*(x) \left( \frac{\det \sigma_{F_{\rho,\eta}}(T_{\rho,\eta}x)}{\det \sigma_F(x)} \right)^{1/2} Y_\rho^\eta(x) da^{Z_u}, \quad (4.32)
\end{aligned}$$

which proves the following theorem.

**Theorem 4.7** *Let  $F \in D_{1+\epsilon}^{\infty-}$  satisfy the conditions of Theorem 4.1 and  $\eta \in D_{1+\epsilon}^{\infty-}(X; H)$  be such that*

- $(I + \rho\eta)$  is one-to-one and onto when restricted to a domain  $B_\eta$  with complement having  $C_\epsilon^p$  capacity 0 for all  $p$ ;
- $(I_H + \rho\nabla\eta)$  is an invertible operator on  $H$ , when restricted to  $B_\eta$ .

Then,

$$\begin{aligned} \frac{da^{Z_u} \circ T_{\rho,\eta}}{da^{Z_u}}(x) &= \left( \frac{\det \sigma_{F_{\rho,\eta}}(T_{\rho,\eta}x)}{\det \sigma_F(x)} \right)^{1/2} Y_\rho^\eta(x) \\ &\stackrel{\Delta}{=} J_{\rho,\eta}^{Z_u}. \end{aligned} \quad (4.33)$$

**Remark 4.8** 1. In order to better understand the above theorem, we shall now try to simplify the expression involved in (4.33), for the simple case where  $F = \delta(h)$ , for some  $h \in H$ , and  $\eta = \nabla F = h$ , is a constant vector field. Clearly,  $\nabla\eta \equiv 0$  implying  $DT_\rho = DT_\rho^{-1} = I_H$ . Then the whole expression boils down to

$$\begin{aligned} J_{\rho,\eta}^{Z_u} &= Y_\rho^\eta(x) \\ &= \exp(-\rho\delta(\eta) - \rho^2\|\eta\|_H^2/2). \end{aligned}$$

2. The above expression in (4.33) can be rewritten as

$$J_{\rho,\eta}^{Z_u} = \left( \frac{\det(\langle DT_{\rho,\eta}^{-1}(x)\nabla F_i(x), DT_{\rho,\eta}^{-1}(x)\nabla F_j(x) \rangle_H)_{ij}}{\det(\langle \nabla F_i(x), \nabla F_j(x) \rangle_H)_{ij}} \right)^{1/2} Y_\rho^\eta(x),$$

where  $DT_\rho^{-1}$  is the operator given by  $(I_H + \rho\nabla\eta)^{-1}$ .

3. From the definition of  $Y_\rho^\eta(x)$ , note that the expression in (4.33) is well defined as long as  $\nabla\eta$  is a Hilbert–Schmidt class valued operator acting on  $H \times H$ . Next, in order to be able to use the formula in (4.33), we need  $J_{\rho,\eta}^{Z_u}$  to be integrable with respect to the surface measure  $da^{Z_u}$ . These are precisely the conditions needed in Theorem 4.3. Later, in Section 5, we shall notice that due to a specific choice of  $F$ , these condition boils down to the integrability of  $Y_\rho^\eta(x)$  with respect to the surface measure  $da^{Z_u}$ .
4. Clearly, the submanifold  $Z_u$ , and the corresponding surface measure  $da^{Z_u}$  are not dependent on  $F$ , and can also be represented by some other functionals, thereby implying that the above expression is independent of the choice of  $\nabla F_i$ . Therefore, to make the above calculation simpler we can choose an appropriate functional  $F'$ , such that  $\{F'(x) = \mathbf{v}\} = Z_u$ , for some  $\mathbf{v} \in \mathbb{R}^k$ , and that  $\{\nabla F'_i\}$  form an orthonormal basis of the normal space of  $Z_u$ .
5. Finally, we note that  $a^{Z_u} \circ T_{\rho,\eta}$  is defined only up to capacity  $C_\epsilon^p$  sets for any  $p$ . That is,  $C_\epsilon^p(A) = 0$  implies  $a^{Z_u} \circ T_{\rho,\eta}(A) = 0$ . Hence, the image of the discontinuities of  $Z_u$  under  $T_{\rho,\eta}$  has  $C_\epsilon^p$  capacity 0. Note the loss of one order of differentiability in the capacity.



### 4.3 The set and its tube

Finally, we shall define the class of sets for which we shall prove a tube formula, and therefore, define the GMFs. In our bid to keep the calculations much easier to handle, we shall restrict our attention to the unit codimensional case.

As can be seen in the previous sections, we have been defining the subsets of the Wiener space via Wiener functionals. Continuing the trend, we shall start with a non-degenerate, Wiener functional  $F \in D_{2+\epsilon}^{\infty-}(X; \mathbb{R})$ , such that  $F$  is an  $H$ -convex functional.

We shall write  $A_u = F^{-1}(-\infty, u]$  for  $u \in \mathcal{O}$ . This is an  $H$ -convex set, and its boundary  $\partial A_u$  is a smooth unit codimensional submanifold of the Wiener space.

The sets like  $A_u$ , shall precisely be sets of our interest throughout the rest of this paper. We shall note here that, in finite dimensions, tube formulae are valid for sets with positive reach, and it is reasonable to conjecture that given an appropriate generalization of the definition of *reach* of an infinite dimensional set, such results may still be true in infinite dimensions. Whereas, for the sake of simplicity, we shall restrict our attention to  $H$ -convex sets.

#### 4.3.1 Description of the tube

The local structure of the tube is determined by the local structure of  $A_u$ . For each  $x \in A_u$  define the support cone

$$\mathcal{S}_x(A_u) = \{h \in H : \text{for any } \delta > 0, \exists 0 < \varepsilon < \delta \text{ such that } x + \varepsilon h \in A_u\}$$

and its dual, the (convex) normal cone

$$N_x(A_u) = \left\{ h \in H : \langle h, h' \rangle \leq 0 \ \forall h' \in \overline{\mathcal{S}_x(A_u)} \right\}$$

where  $\overline{\mathcal{S}_x(A_u)}$  is the closure of  $\mathcal{S}_x(A_u)$  in  $H$ .

The following lemmas describe some of the properties of the tube around  $A_u$ . For clarity we state the results only for the case of codimension 1, though similar statements hold for codimension  $k$ .

Define the smooth points of  $F$

$$\begin{aligned} \text{Sm}(F) \triangleq & \left\{ x : \nabla F(x) \neq 0, \forall h_1, h_2 \in H_\infty \exists \varepsilon_n(h_1, h_2), \downarrow 0 \right. \\ & \left. \text{such that } \lim_{n \rightarrow \infty} T_2(F, x, h_1, h_2, \varepsilon_n) = 0 \right\} \end{aligned} \quad (4.34)$$

where

$$\begin{aligned} T_2(F, x, h_1, h_2, \varepsilon) = & F(x + h_1) + \varepsilon \langle \nabla F(x + h_1), h_2 \rangle_H + \\ & \varepsilon^2 \langle \nabla^2 F(x + h_1), h_2 \otimes h_2 \rangle_{H \otimes H} - F(x + h_1 + \varepsilon h_2) \end{aligned}$$

is the difference between the second-order Taylor expansion of  $F(x + h_1 + \varepsilon h_2)$  evaluated at  $x + h_1$  and its true value.

**Lemma 4.9** Suppose  $F \in D_{2+\epsilon}^{\infty-}$ . Then,

$$C_\epsilon^p(\text{Sm}(F)^c) = 0, \quad \forall p.$$

At every  $x \in \partial A_{\mathbf{u}} \cap \text{Sm}(F)$ ,

$$N_x(A_{\mathbf{u}}) = \{c\nabla F(x) : c \geq 0\}. \quad (4.35)$$

**Proof:** The first conclusion follows essentially directly from Corollary 3.11. Suppose now that  $x \in \text{Sm}(F)$ . Then, for any  $h \perp \nabla F(x)$ ,  $\|h\| \leq K$  we can find a sequence  $H_\infty \ni h_n \xrightarrow{n \rightarrow \infty} h$  satisfying  $\langle h_n, \nabla F(x) \rangle_H < -1/n$ . Because the 2nd order Taylor expansion holds at  $x$  we see that  $h_n \in \mathcal{S}_x(A_{\mathbf{u}})$  for all  $n$ . Hence,  $h \in \overline{\mathcal{S}_x(A_{\mathbf{u}})}$ . This is enough to conclude that any  $\eta \in N_x(A_{\mathbf{u}})$  is parallel to  $\nabla F(x)$ . It is not hard to see that it must therefore be a positive multiple of  $\nabla F(x)$ .  $\square$

**Lemma 4.10** Suppose  $F \in D_{2+\epsilon}^{\infty-}$  is  $H$ -convex. Then, for each  $r > 0$ , the restriction of

$$x \mapsto x + r\nabla F(x)/\|\nabla F(x)\| \triangleq x + r\eta_x$$

to  $\text{Sm}(F) \cap \partial A_{\mathbf{u}}$  is one-to-one in the sense that for each  $x \in \text{Sm}(F) \cap \partial A_{\mathbf{u}}$

$$\{y \in \text{Sm}(F) \cap \partial A_{\mathbf{u}} : \|y - (x + r\eta_x)\|_H \leq r\} = \emptyset.$$

**Proof:** Given  $x \in \text{Sm}(F) \cap \partial A_{\mathbf{u}}$ , suppose such a  $y$  exists with  $\|x + r\eta_x - y\| < r$ . As  $y$  is a smooth point of  $F$ , we can find some  $h \in H_\infty$  such that  $\|x + r\eta_x - (y + h)\|_H < r$  and  $F(y + h) < u$  with  $F$  continuous at  $y + h$ . Choose  $\nu(x, y) \in H_\infty$  such that  $x + \nu(x, y)$  is arbitrarily close to  $y + h$ . Then, by continuity of  $F$  on  $\text{Sm}(F)$ ,  $F(x + \nu(x, y)) < u$  and  $\|x + r\eta_x - (x + \nu(x, y))\|_H = \|r\eta_x - \nu(x, y)\|_H < r$ . Note that this implies  $\langle \nu(x, y), \eta_x \rangle_H > 0$ , or, alternatively  $\nu(x, y) \notin \mathcal{S}_x(A_{\mathbf{u}})$ .

Now consider the restriction of  $F$  to the line segment joining  $[x, x + \nu(x, y)]$ , denoted by

$$f(t) = F(x + t(x - \nu(x, y))), \quad 0 \leq t \leq 1$$

which, by Remark 3.13 is continuous, twice-differentiable and convex on a dense subset of  $t \in [0, 1]$  hence we can find a continuous, twice-differentiable convex function  $\tilde{f}$  on all of  $[0, 1]$  that agrees with  $f$  on this dense subset.

There are two possibilities, the first being that  $\tilde{f}(t) \leq u$  for all  $t \in [0, 1]$ . This would imply  $\eta(x, y) \in \mathcal{S}_x(A_{\mathbf{u}})$  contradicting our previous observation. The second alternative is that there exists  $t$  such that  $\tilde{f}(t) > u$ . However,  $\tilde{f}(0) = u$ ,  $\tilde{f}(1) < u$  and this would violate convexity. By contradiction, there can be no such  $y$ .

This proves the assertion that there are no points  $y$  of distance strictly less than  $r$  to  $x + r\eta_x$ . Now, suppose there exists a smooth point  $y \neq x$  of distance exactly  $r$  from  $x + r\eta_x$ . Then, for any  $\delta > 0$  it is not hard to show that

$$\|y - (x + (\delta + r)\eta_x)\|_H < \delta + r$$

but we just proved that there can be no such  $y$ .  $\square$

We are now in a position to define the tube

$$\begin{aligned} \text{Tube}(A_u, \rho) &= \{y \in X : \exists x \in A_u, \|y - x\|_H \leq \rho\} \\ &= \{y \in X : d_H(y, A_u) \leq \rho\} \end{aligned} \quad (4.36)$$

where the distance function is defined as

$$d_H(y, A_u) = \inf_{h \in H_\infty : y - h \in A_u} \|h\|_H. \quad (4.37)$$

The level sets of the distance function are hypersurfaces at distance  $r$

$$\partial A_u^r = \{y \in X : d_H(y, A_u) = r\}. \quad (4.38)$$

Lemma 4.10 asserts that the restriction of

$$x \mapsto x + r\eta_x$$

to  $A_u \cap \text{Sm}(F)$  is one-to-one. On the image of  $\text{Sm}(F)$ , its inverse is easily defined

$$x + r\eta_x \mapsto (x, \eta_x)$$

and, as noted in the remarks following Theorem 4.7 the image of  $\partial A_u \cap \text{Sm}(F)$  has  $C_\epsilon^p$ -capacity 0. Hence, up to a set of  $C_\epsilon^p$ -capacity 0, it is a bijection and Theorem 4.7 can be applied to study the surface measure of  $\partial A_u^r$ .

Moreover, the following theorem further corroborates the fact that the change of measure formula established in Theorem 4.7 is the appropriate result to use in order to obtain a tube formula, as will be seen later.

**Theorem 4.11** *Let  $C_{\epsilon_1}^{\infty-}(A) = 0$ , then under hypotheses (H2) and (H3) of [13], for  $\epsilon_1 < \epsilon$*

$$C_{\epsilon_1}^{\infty-}(A + B_H(0, r)) = 0,$$

where  $B_H(0, r)$  is a ball in  $H$  centered at 0 with radius  $r$ .

Since, capacities are continuous from below, it suffices to prove that  $C_{\epsilon_1}^{\infty-}(A \oplus B_{E_n}(0, r)) = 0$ , for each  $n$ , whenever  $C_{\epsilon_1}^{\infty-}(A) = 0$ , where  $B_{E_n}(0, r)$  is a ball of radius  $r$ , centered at 0, in the vector space  $E_n = \text{span}(h_1, \dots, h_n)$ , where  $\{h_i\}_{i \geq 1}$  is the orthonormal basis of  $H$ . Also, note that the proof is given for an open subset  $A$  of the Wiener space  $X$ , but using the arguments of [13], we can extend it to general subsets of the Wiener space.

Before embarking on proving the above theorem, we shall, first, obtain some estimates on functionals derived from the Wiener functionals.

Let  $e_A$  be the potential equilibrium of  $A$ , and

$$A \oplus B_{E_n}(0, r) = \{(A + \langle s, h^{(n)} \rangle) : s \in B_{\mathbb{R}^n}(0, r)\},$$

where  $\langle s, h^{(n)} \rangle = \sum_{i=1}^n s_i h_i$ . Further for later part, we shall denote  $I_n \subset B_{\mathbb{R}^n}(0, r)$  as the set of all rationals in the set  $B_{\mathbb{R}^n}(0, r)$ .

We shall start with a technical result, which is, essentially, an extension of Theorem 2.1 of [12], and is stated below<sup>10</sup>.

**Theorem 4.12** *Let  $f \in D_\alpha^p(X)$  for  $\alpha \in (1/p, 1)$ , and  $\mathbb{R}^n \ni t \mapsto \xi(t, \cdot) = f(\cdot + \langle t, h^{(n)} \rangle)$ , such that  $|t| \leq T$ , for some fixed  $T$  i.e.,  $t$  belongs to some large enough cube. Then for all  $p' \in (1/\alpha, p)$  there exists a  $C = C(p, p', \alpha, T)$ , such that*

$$\|\xi(t) - \xi(s)\|_{p'} \leq C \|f\|_{p, \alpha} |t - s|^\alpha.$$

**Proof:** Before we shall start proving the above result, we shall recall that the estimates of Lemma 4.1 of [12] remain unchanged in our setup. Now we need an estimate analogous to the one obtained in Lemma 4.2 of [12], for which we recall the Ramer's change of measure formula,

$$\begin{aligned} & \|G(\cdot + \langle t_2, h^{(n)} \rangle) - G(\cdot + \langle t_1, h^{(n)} \rangle)\|_{p'} \\ &= \|G(\cdot + \frac{1}{2}\langle t_1 + t_2, h^{(n)} \rangle) + \frac{1}{2}\langle t_2 - t_1, h^{(n)} \rangle) - G(\cdot + \frac{1}{2}\langle t_1 + t_2, h^{(n)} \rangle) - \frac{1}{2}\langle t_2 - t_1, h^{(n)} \rangle)\|_{p'} \\ &= \left( \int_X |G(x + \frac{1}{2}\langle t_2 - t_1, h^{(n)} \rangle) - G(x - \frac{1}{2}\langle t_2 - t_1, h^{(n)} \rangle)|^{p'} \right. \\ & \quad \left. \exp(-\frac{1}{2} \|\frac{1}{2}\langle t_1 + t_2, h^{(n)} \rangle\|_H^2 - \delta(\frac{1}{2}\langle t_1 + t_2, h^{(n)} \rangle) \mu(dx)) \right)^{1/p'} \end{aligned} \quad (4.39)$$

Now writing  $h_1 = \frac{1}{|t_2 - t_1|} \langle t_2 - t_1, h^{(n)} \rangle$ ,  $h_2 = \frac{1}{|t_1 + t_2|} \langle t_1 + t_2, h^{(n)} \rangle$ , and  $Y_{|t_1 + t_2|/2}^{h_2} = \exp[-\frac{1}{2} \|\frac{|t_1 + t_2|}{2} h_2\|_H^2 - \delta(\frac{|t_1 + t_2|}{2} h_2)]$ , we can rewrite the above as

$$\begin{aligned} & \|G(\cdot + \langle t_2, h^{(n)} \rangle) - G(\cdot + \langle t_1, h^{(n)} \rangle)\|_{p'} \\ &= \left( \int_X |G(x + \frac{1}{2}|t_2 - t_1|h_1) - G(x - \frac{1}{2}|t_2 - t_1|h_1)|^{p'} \times Y_{|t_1 + t_2|/2}^{h_2}(x) \mu(dx) \right)^{1/p'}, \end{aligned} \quad (4.40)$$

This reduces the above expression to the case dealt in [12]. Therefore, using the rest of the calculations of Lemma 4.2 of [12], and writing  $G = T_a f$ , where  $\{T_a\}_{a \geq 0}$  is the semigroup associated with the Ornstein–Uhlenbeck operator  $L$ , we get the desired estimate expressed as

$$\|T_a f(\cdot + \langle t_2, h^{(n)} \rangle) - T_a f(\cdot + \langle t_1, h^{(n)} \rangle)\|_{p'} \leq C(p, p', \alpha) \|T_a f\|_{p, 1} |t_2 - t_1|.$$

Thereafter, we can mimic the proof of Theorem 2.1 of [12], and get the desired estimate.  $\square$

Now coming back to our case,  $e_A \in D_\epsilon^{\infty-}$ , therefore there exists a  $v_A \in L^{\infty-}$ , such that

$$e_A = (I - L)^{-\epsilon/2} v_A \triangleq (I - L)^{-\epsilon_1/2} v_{(\epsilon - \epsilon_1), A},$$

---

<sup>10</sup>We shall note here that the assumption (A), needed to prove Theorem 2.1 in [12], holds automatically for our setup, as is noted in Section 3 of [12]

where  $\epsilon_1$  is some number strictly smaller than  $\epsilon$ , and  $L$  is the Ornstein–Uhlenbeck operator. Then, clearly,

$$e_A(\cdot + \langle t, h^{(n)} \rangle) = (I - L)^{-\epsilon/2} v_A(\cdot + \langle t, h^{(n)} \rangle) \triangleq (I - L)^{-\epsilon_1/2} v_{(\epsilon - \epsilon_1), A}(\cdot + \langle t, h^{(n)} \rangle).$$

Now, writing  $\xi(t) \triangleq e_A(\cdot + \langle t, h^{(n)} \rangle)$ , and  $\xi_{(\epsilon - \epsilon_1)}(t) \triangleq v_{(\epsilon - \epsilon_1), A}(\cdot + \langle t, h^{(n)} \rangle)$ , and also, in the process, choosing the appropriate quasi-continuous redefinitions of the processes  $\xi$  and  $\xi_{(\epsilon - \epsilon_1)}$ , and choosing a large  $p'$  (conditions on  $p'$  will appear later), such that By Kree–Meyer inequalities, we have

$$\|\xi(t) - \xi(s)\|_{p', \epsilon_1} \leq C \|\xi_{(\epsilon - \epsilon_1)}(t) - \xi_{(\epsilon - \epsilon_1)}(s)\|_{p'}. \quad (4.41)$$

Now using the above theorem with  $f$  replaced by  $v_{(\epsilon - \epsilon_1), A}$ , we get

$$\begin{aligned} \|\xi_{(\epsilon - \epsilon_1)}(t) - \xi_{(\epsilon - \epsilon_1)}(s)\|_{p'} &= \|v_{(\epsilon - \epsilon_1), A}(\cdot + \langle t, h^{(n)} \rangle) - v_{(\epsilon - \epsilon_1), A}(\cdot + \langle s, h^{(n)} \rangle)\|_{p'} \\ &\leq C \|v_{(\epsilon - \epsilon_1), A}\|_{p, (\epsilon - \epsilon_1)} |t - s|^{(\epsilon - \epsilon_1)} \\ &= C \|e_A\|_{p, \epsilon} |t - s|^{(\epsilon - \epsilon_1)} \end{aligned} \quad (4.42)$$

where  $p' \in (\frac{2}{\epsilon}, p)$ . Combining, (4.41) and (4.42), we get

$$\|\xi(t) - \xi(s)\|_{p', \epsilon_1} \leq C \|e_A\|_{p, \epsilon} |t - s|^{(\epsilon - \epsilon_1)}, \quad (4.43)$$

which can be rewritten as

$$\sup_{s \neq t} \frac{\|\xi(t) - \xi(s)\|_{p', \epsilon_1}^{p'}}{|t - s|^{p'(\epsilon - \epsilon_1)}} \leq C \|e_A\|_{p, \epsilon}^{p'}. \quad (4.44)$$

Now we can list the assumptions on the various indices as follows: we start with any fixed  $\epsilon_1 < \epsilon$ , then choose a large enough  $p$  such that  $(\epsilon - \epsilon_1) \in (1/p, 1)$ , and then we choose  $p'$  such that  $p' \in (1/(\epsilon - \epsilon_1), p)$  and  $p'(\epsilon - \epsilon_1) > n$ . This can be achieved by choosing  $p$  and  $p'$  of the order of  $n$ , in particular, choosing  $p = \frac{an}{(\epsilon - \epsilon_1)}$ , and  $p' = \frac{bn}{(\epsilon - \epsilon_1)}$ , for  $a > b$  will do.

Now using Theorem 3.4 of Shigekawa [14], we get

$$C_{\epsilon_1}^p \left( \sup_{s \neq t} |\xi(t) - \xi(s)| \right) \leq C \|e_A\|_{p, \epsilon}^{p'}. \quad (4.45)$$

**Proof of Theorem 4.11:** Now let us consider

$$\begin{aligned} C_{\epsilon_1}^p (A \oplus B_{E_n}(0, r)) &= C_{\epsilon_1}^p \left( \bigcup_{s \in B_{\mathbb{R}^n}(0, r)} (A + \langle s, h^{(n)} \rangle) \right) \\ &= C_{\epsilon_1}^p \left( \sup_{s \in B_{\mathbb{R}^n}(0, r)} 1_A(\cdot + \langle s, h^{(n)} \rangle) \right) \\ &\leq C_{\epsilon_1}^p \left( \sup_{s \in I_n} e_A(\cdot + \langle s, h^{(n)} \rangle) \right) \quad \text{as } e_A \geq 1_A \\ &\leq C_{\epsilon_1}^p \left( \sup_{s \in B_{\mathbb{R}^n}(0, r)} \xi(s) \right) \\ &\leq C_{\epsilon_1}^p \left( \sup_{s \in B_{\mathbb{R}^n}(0, r)} |\xi(s) - \xi(0)| + |\xi(0)| \right) \end{aligned} \quad (4.46)$$

Now using (4.45), we shall get

$$\begin{aligned}
C_{\epsilon_1}^p \left( A \oplus B_{E_n}(0, r) \right) &\leq C_{\epsilon_1}^p \left( \sup_{s \in B_{\mathbb{R}^n}(0, r)} |\xi(s) - \xi(0)| + |\xi(0)| \right) \\
&\leq (C + 1) \|e_A\|_{p, \epsilon}^{p'} \\
&= (C + 1) (C_\epsilon^p(A))^{\frac{p'}{p}},
\end{aligned}$$

which proves that  $C_{\epsilon_1}^{\infty-}(A \oplus B_{E_n}(0, r)) = 0$ , for all  $p > n(\epsilon - \epsilon_1)^{-1}$ . Now by the definition of the capacities and the hierarchy of the Sobolev spaces, we shall have  $C_{\epsilon_1}^{\infty-}(A \oplus B_{E_n}(0, r)) = 0$ . Thereby proving the result.  $\square$

Using the definition of the smooth points  $\text{Sm}(F)$ , and the tube  $\text{Tube}(A_u, \rho)$ , we can conclude that,

$$\begin{aligned}
\text{Tube}(A_u, \rho) &= A_u \oplus B_H(0, \rho) \\
&= ((A_u \cap \text{Sm}(F)) \oplus B_H(0, \rho)) \\
&\quad \cup ((A_u \cap \{\text{Sm}(F)\}^c) \oplus B_H(0, \rho))
\end{aligned} \tag{4.47}$$

But, using by the above calculations, we have

$$\mu(\text{Tube}(A_u, \rho)) = \mu((A_u \cap \text{Sm}(F)) \oplus B_H(0, \rho)), \tag{4.48}$$

since,  $C_{\epsilon_1}^p((A_u \cap \{\text{Sm}(F)\}^c) \oplus B_H(0, \rho)) = 0$ , implying that the  $\mu$ -measure of the set is zero. Therefore, it is enough, for the tube formula, to consider the set  $((A_u \cap \text{Sm}(F)) \oplus B_H(0, \rho))$ , on which the transformation  $x \mapsto x + \eta_x$  is well defined upto  $C_\epsilon^p$ -zero sets, and hence we can use the change of measure formula for the surface areas given in Theorem 4.7.

## 5 A Wiener tube formula

After setting up the basics, definitions and the conditions, concerning a tube formula in the Wiener space, we shall finally prove one of the main result of this paper, which can be stated in the form of the following theorem.

**Theorem 5.1** *Let  $F \in D_{2+\delta}^{\infty-}(X; \mathbb{R})$  be an  $H$ -convex Wiener functional such that it satisfies all the regularity conditions of Theorem 4.3, and  $A_u = F^{-1}(-\infty, \mathbf{u}]$ , then*

$$\mu(\text{Tube}(A_u, \rho)) = \mathcal{M}_0^\mu(A_u) + \sum_{j=1}^{\infty} \frac{\rho^j}{j!} \mathcal{M}_j^\mu(A_u),$$

where  $\mathcal{M}_j^\mu(A_u)$  are the infinite dimensional versions of Gaussian Minkowski functionals, and as usual  $\mathcal{M}_0^\mu(A_u) = \mu(A_u)$ .

**Proof:** Let us start with recalling the definition of the *outward* pointing normal space  $N_x(A_u)$  from (4.35), and writing  $N(A_u) = \cup_{x \in A_u} N_x(A_u)$ . Then, let us

define a distance function,  $d_{A_u} : \text{Tube}(A_u, \rho) \rightarrow \mathbb{R}$ , such that for  $x \in \text{Tube}(A_u, \rho)$  writing the “residual” as

$$\hat{r}_x = \underset{r \in \mathbb{R}; \eta \in N(A_u)}{\operatorname{argmin}} d(x - r\eta, A_u),$$

the distance function  $d_{A_u}$  is given by

$$d_{A_u}(x) = \|\hat{r}_x\|.$$

Clearly, from the above definition,  $d_{A_u}^{-1}(0) = A_u$ . Also, we can further express  $\text{Tube}(A_u, \rho)$  as the disjoint union of  $A_u$  and  $\text{Tube}^+(\partial A_u, \rho)$ , where  $\text{Tube}^+(\partial A_u, \rho) = \text{Tube}(\partial A_u, \rho) \cap A_u^c$ . Thus,

$$\mu(\text{Tube}(A_u, \rho)) = \mu(A_u) + \mu(\text{Tube}^+(\partial A_u, \rho)) \quad (5.49)$$

Now using the Wiener space version of Federer’s co-area formula as it appears in [3], we shall obtain

$$\begin{aligned} \mu(\text{Tube}^+(\partial A_u, \rho)) &= E(1_{\text{Tube}^+(\partial A_u, \rho)}) \\ &= \int_0^\rho \int_{d_{A_u}^{-1}(r)} (\sigma_{d_{A_u}}(x))^{-1} da^{\partial^+ A_u^r} dr, \end{aligned} \quad (5.50)$$

where,  $\partial^+ A_u^r = d_{A_u}^{-1}(r) \cap A_u^c$ , are the level sets of the distance function  $d_{A_u}$  in the outward direction.

Now note that,  $\nabla d_{A_u} = \eta$ , hence  $\sigma_{d_{A_u}}(x) = 1$ . Then let us define the transformation  $T_{r,\eta} : X \rightarrow X$ , such that its restriction to  $A_u$  is given by

$$T_{r,\eta}(x) = x + r\eta.$$

Clearly,  $T_{r,\eta}(\partial A_u) = \partial A_u^r$ . Then, we shall use our change of measure formula for surfaces on  $\int_{d_{A_u}^{-1}(r)}$ , to further simplify the expression in (5.50) to obtain

$$\begin{aligned} \mu(\text{Tube}^+(\partial A_u, \rho)) &= \int_0^\rho \int_{A_u} J_{r,\eta}^{\partial A_u} da^{\partial A_u} dr \\ &= \int_0^\rho \int_{A_u} Y_r^\eta da^{\partial A_u} dr \end{aligned}$$

where terms  $J_{r,\eta}^{\partial A_u}$  and  $Y_r^\eta$  are as they appear in Theorem 4.7.

Now using a Taylor series expansion for  $Y_r^\eta$  with respect to  $r$ , we can rewrite the above expression as

$$\begin{aligned} \mu(\text{Tube}^+(\partial A_u, \rho)) &= \sum_{j=0}^{\infty} \int_0^\rho \int_{A_u} \frac{r^j}{j!} \frac{d^j}{dr^j} \left( \det_2(I_H + r\nabla\eta) \exp(-r\delta(\eta) - r^2/2) \right) \Big|_{r=0} da^{\partial A_u} dr \\ &= \sum_{j=0}^{\infty} \frac{\rho^{j+1}}{(j+1)!} \int_{A_u} \frac{d^j}{dr^j} \left( \det_2(I_H + r\nabla\eta) \exp(-r\delta(\eta) - r^2/2) \right) \Big|_{r=0} da^{\partial A_u}. \end{aligned} \quad (5.51)$$

We note here that  $\rho$  must be within the radius of convergence of the Taylor series of  $Y_r^\eta$ , which in turn will ensure the convergence of the above series.

Finally, plugging the above expression in (5.49), we get,

$$\begin{aligned} & \mu(\text{Tube}(A_{\mathbf{u}}, \rho)) \\ &= \mu(A_{\mathbf{u}}) + \sum_{j=1}^{\infty} \frac{\rho^j}{j!} \int_{A_{\mathbf{u}}} \frac{d^j}{dr^j} \left( \det_2(I_H + r \nabla \eta) \exp(-r\delta(\eta) - r^2/2) \right) \Big|_{r=0} da^{\partial A_{\mathbf{u}}}. \end{aligned} \quad (5.52)$$

The above expression can be rewritten as,

$$\mu(\text{Tube}(A_{\mathbf{u}}, \rho)) = \mu(A_{\mathbf{u}}) + \sum_{n=1}^{\infty} \frac{\rho^n}{n!} \mathcal{M}_n^\mu(A_{\mathbf{u}}),$$

where  $\mathcal{M}_n^\mu(A_{\mathbf{u}})$  are Gaussian Minkowski functionals of the infinite dimensional set  $A_{\mathbf{u}}$ , given by

$$\mathcal{M}_n^\mu(A_{\mathbf{u}}) = \int_{A_{\mathbf{u}}} \frac{d^n}{dr^n} \left( \det_2(I_H + r \nabla \eta) \exp(-r\delta(\eta) - r^2/2) \right) \Big|_{r=0} da^{\partial A_{\mathbf{u}}}. \quad (5.53)$$

which proves the theorem.  $\square$

**Remark 5.2** Let  $F = (F_1, \dots, F_k) \in D_{2+\epsilon}^{\infty-}(X; \mathbb{R}^k)$ , such that each  $F_i$  is an  $H$ -convex Wiener functional. Then a similar tube formula can be proven for higher codimensional sets of the form  $A_{\mathbf{u}} = \cap_{1 \leq i \leq k} F_i^{-1}(-\infty, u_i)$  for  $\mathbf{u} = (u_1, \dots, u_k) \in \mathcal{O}$ , with appropriate modifications to the definition of Gaussian Minkowski functionals.

## 6 Applications

In this section, we shall invoke the existential results from the previous section to obtain, a kinematic fundamental formula similar to the one obtained in Theorem 15.9.5 of [2], though, for a larger class of random fields.

To start with, let us consider a real valued random field  $f$  defined on a compact Riemannian manifold  $M$ , equipped with a metric  $\tau$ . Then the modulus of continuity  $\Xi$ , of a function  $F : M \rightarrow \mathbb{R}$ , is defined as

$$\Xi_F(\eta) \triangleq \sup_{\tau(x,y) \leq \eta} |F(x) - F(y)|,$$

for all  $\eta > 0$ .

Continuing the setup introduced in the examples stated in Section 1, we shall consider a specific class of random fields  $f$  which can be represented as

$$f(x) = \sum_{i=1}^N \int_0^1 V_i(B_i^x(s)) dB_i^x(s), \quad (6.54)$$



where the integral is to be interpreted in the  $It\hat{o}$  sense, and each  $V_i : \mathbb{R} \rightarrow \mathbb{R}$  are smooth functions, and  $B^x(s) = (B_i^x(s))_{i=1}^N$  is  $\mathbb{R}^N$  valued, zero-mean Gaussian process whose covariance is given by

$$E(B_i^x(s)B_j^y(t)) = (s \wedge t) C_{ij}(x, y) = (s \wedge t) C(x, y), \quad (6.55)$$

where  $C : M \times M \rightarrow \mathbb{R}$  is a smooth function, such that for each fixed  $t \in [0, 1]$ , the field  $B^\cdot(t)$  is an isotropic<sup>11</sup> Gaussian field over  $M$  (see Sections 5.7 and 5.8 of [2]).

The spatial derivative of such a random field  $f$  is given by,

$$\nabla f(x) = \sum_{i=1}^N \int_0^1 \nabla V_i(B_i^x(s)) \nabla B_i^x(s) dB_i^x(s) + \sum_{i=1}^N \int_0^1 V_i(B_i^x(s)) d\nabla B_i^x(s).$$

Similarly, we can obtain expressions for derivatives of higher order.

Before we embark upon proving the main result of this section, we shall impose some conditions on the functions  $V_i$  and the Gaussian process  $B^x(t)$ .

- (A1) All the  $V_i$ ,  $1 \leq i \leq N$  are  $C^4$ , which is the class of all 4-continuously differentiable functions.
- (A2) Writing  $V^{(k)}$  as the  $k$ -th derivative of  $V$  for  $k \geq 1$ , let us define

$$C_i(k, s, x, y) \triangleq \sup_{0 \leq \alpha \leq 1} V_i^{(k)}(\alpha B^x(s) + (1 - \alpha)B^y(s)),$$

for any  $x, y \in M$  and  $k = 0, 1, 2, 3, 4$ . Then, for some  $p \gg \dim(M)$  and for all  $i = 1, \dots, N$ ,

$$\sup_{0 \leq s \leq 1} \|C_i(k, s, x, y)\|_p^p = c_{i,k}(x, y, p) < \infty.$$

Also,  $\sup_{x \neq y} c_{i,k}(x, y, p) < \infty$ , for all  $k = 0, 1, 2, 3, 4$ . Note that this is satisfied whenever the  $V_i$ 's are  $C^4$  with polynomial growth.

- (A3) For each  $r \geq 1$ , there exists a constant  $m_r$ , such that

$$E|B^x(s) - B^y(s)|^r \leq m_r |x - y|^r, \quad \forall 0 \leq s \leq 1,$$

where  $m_r$  depends solely on  $r$ .

- (A4) All the above assumptions also hold true with  $B^x(s)$  replaced by  $\nabla B^x(s)$  and  $\nabla^2 B^x(s)$ , respectively.

Next, we recall the definition of an *excursion set*  $A_u$  corresponding to a field  $f : M \rightarrow \mathbb{R}$ , as

$$A_u(f; M) \triangleq \{x \in M : f(x) \geq u\}.$$

Also, note that writing  $F(\omega) = \int_0^1 V(\omega_s) d\omega_s$ , one can consider the random field defined above as  $f(x) = F(B^x)$ .

<sup>11</sup>To see how isotropy makes the calculations much simpler, we refer the reader to Section 12.5 of [2].

**Theorem 6.1** *Let  $M$  be a  $m$ -dimensional manifold, and  $f$  be a random field defined on  $M$ , and represented as in (6.54), and satisfying the conditions (A1) – (A4). Also, let  $f(x)$  and  $\nabla f(x)$  be nondegenerate in the sense of Malliavin, for some  $x \in M$ <sup>12</sup>, and that the corresponding Wiener functional  $F$ , satisfies the exponential moment condition specified in Theorem 4.3. Then writing  $A_u(f)$  as the excursion set for the random field  $f$ , and  $\mathcal{L}_i(\cdot)$  as the  $i$ -th Lipschitz-Killing curvature,*

$$E(\mathcal{L}_0(A_u(f; M))) = \sum_{j=0}^m (2\pi)^{-j/2} \mathcal{L}_j(M) \mathcal{M}_j^\mu(F^{-1}[u, \infty)), \quad (6.56)$$

where  $F^{-1}([u, \infty))$  is a subset of the Wiener space  $X$  with  $\mathcal{M}_j^\mu$  as its GMF, as defined in the previous section.

Before we start proving the theorem, we shall try to argue in favor of the above expression. Note that the LHS of (6.56) is *additive* in the index  $M$ , and that it is invariant under rigid transformations of  $M$ . Now we shall recall one of the best known results in integral geometry by Hadwiger, which characterizes such set functionals, and the proof of which can be found in [6].

**Lemma 6.2** *Let  $\psi$  be a real valued function defined on **basic complexes**<sup>13</sup> in  $\mathbb{R}^K$ , invariant under rigid motions, additive, in the sense of measures, and monotone, in the sense that for all pairs  $A, B$ , either  $A \subseteq B \Rightarrow \psi(A) \leq \psi(B)$  or  $A \subseteq B \Rightarrow \psi(A) \geq \psi(B)$ . Then*

$$\psi(A) = \sum_{j=0}^K a_j \mathcal{L}_j(A),$$

where  $a_0, \dots, a_K$  are nonnegative ( $\psi$ -dependent) constants.

Clearly, the mapping  $M \mapsto E(\mathcal{L}_0(A_u(f, M)))$ , seems a reasonable candidate to apply the above lemma, if only we could prove that the mapping is monotone. However, since we have not established monotonicity (in  $M$ ) of  $E(\mathcal{L}_0(A_u(f, M)))$ , therefore, we shall use a different technique to establish the same result, namely that an approximation to the LHS of (6.56) has the form of the RHS, and that this approximation of the RHS indeed converges to the RHS.

At this point, we note that all the results obtained below are for the case of  $N = 1$ , whereas the using the similar methods, same results are true for general  $N$ . We shall start proving Theorem 6.1 by first listing some regularity properties of field  $f$  defined in (6.54), in the form of the following theorem.

**Theorem 6.3** *Let the random field  $f$  be as defined in (6.54), such that it also satisfies (A1) – (A4), then*

<sup>12</sup>For an isotropic random field, nondegeneracy of the field at some point on the manifold is equivalent to the nondegeneracy of the random field at all points on the manifold.

<sup>13</sup>For precise background and definitions concerning basic complexes, we refer the reader to [2].

- (a)  $F \in D_3^{\infty-}(X; \mathbb{R})$ , and under the assumption of nondegeneracy of  $F$ , the density  $p_F$  of  $F$  is bounded,
- (b)  $f$  is continuous, and that for any  $\epsilon > 0$

$$P(\Xi_f(\eta) > \epsilon) = o(\eta^{\dim(M)}) \text{ as } \eta \downarrow 0,$$

Also, same is true for  $\nabla f$  and  $\nabla^2 f$ .

**Proof:** Recall that  $F = \int_0^1 V(B(t)) dB(t)$ , and  $DF \in H$ , the Cameron–Martin space corresponding to  $C_0[0, 1]$ . Therefore, by definition, there exists a unique  $(\dot{D}F) \in L^2([0, 1])$  such that  $D_r F \triangleq (DF)(r) = \int_0^r (\dot{D}_s F) ds$ , where  $\dot{D}F$  is the time derivative of  $DF$ . Using this notation we shall have,

$$(\dot{D}_r F) = V(B(r)) + \int_0^1 V^{(1)}(B(t)) 1_{[0, t]}(r) dB(t),$$

$$\begin{aligned} (D_s(\dot{D}_r F)) &= V^{(1)}(B(r)) 1_{[0, r]}(s) + V^{(1)}(B(s)) 1_{[0, s]}(r) \\ &\quad + \int_0^1 V^{(2)}(B(t)) 1_{[0, t]}(r) 1_{[0, t]}(s) dB(t) \end{aligned}$$

Clearly, due to the moment conditions imposed on  $V$  and its derivatives, we can conclude that  $F \in D_3^{\infty-}(X; \mathbb{R})$ , and the boundedness of the density  $p_F$  follows using Proposition 2.1.1 of [9].

Now to prove continuity of  $f$  and its derivatives, we shall use Kolmogorov’s continuity criterion. Note that, although, Kolmogorov’s continuity criterion is usually stated for processes with Euclidean parameter space, but since it is a local result, thus, it can easily be extended to processes defined on smooth Riemannian manifolds, as locally (using the charts), the manifolds are Euclidean. Therefore, we present the proof of continuity related results for the field  $f \circ \phi^{-1}$  where  $\phi$  is the local chart<sup>14</sup>, but we shall suppress the chart map, and will write  $f$  for both the field  $f$ , and its counterpart<sup>15</sup>  $f \circ \phi^{-1}$ .

In order to use Kolmogorov’s continuity theorem, we must obtain  $L^p$  estimates for  $(f(x) - f(y))$ . Writing  $V^*$  as any antiderivative of  $V$ , we have

$$V^*(B^x(1)) = V^*(B^x(0)) + \int_0^1 V(B^x(s)) dB^x(s) + \int_0^1 V^{(1)}(B^x(s)) ds.$$

<sup>14</sup>Charts for an  $m$ -dimensional manifold  $M$  are a collection of maps  $\{\phi_j\}$  together with an open cover of the manifold  $M$ , such that  $\phi_j : U_j \rightarrow \mathbb{R}^m$ , together with some consistency conditions.

<sup>15</sup>This notation is unlikely to cause any confusion as smoothness of  $f$  implies smoothness of  $f \circ \phi^{-1}$ , and vice versa, as long as the chart  $\phi$  is *smoother* than  $f$ .

Thus, for  $p \geq 1$ , there exist  $m_{1,p}$  and  $m_{2,p}$  such that

$$\begin{aligned}
& \|f(x) - f(y)\|_p^p \\
&= E|f(x) - f(y)|^p \\
&\leq m_{1,p} E|V^*(B^x(1)) - V^*(B^y(1))|^p + m_{2,p} E \left| \int_0^1 (V^{(1)}(B^x(s)) - V^{(1)}(B^y(s))) ds \right|^p \\
&\leq m_{1,p} E \left| \sup_{\alpha} V[\alpha B^x(1) + (1-\alpha)B^x(1)] \times (B^x(1) - B^y(1)) \right|^p \\
&\quad + m_{2,p} \int_0^1 E|V^{(1)}(B^x(s)) - V^{(1)}(B^y(s))|^p ds \\
&\leq m_{1,p} E|C(0, 1, x, y)(B^x(1) - B^y(1))|^p \\
&\quad + m_{2,p} \int_0^1 E|C(2, s, x, y)(B^x(s) - B^y(s))|^p ds.
\end{aligned}$$

Now choosing  $p_1, p_2 > 0$  such that  $p_1^{-1} + p_2^{-1} = p^{-1}$ , we get

$$\begin{aligned}
& \|f(x) - f(y)\|_p^p \\
&\leq m_{1,p} (\|C(0, 1, x, y)\|_{p_1} \|B^x(1) - B^y(1)\|_{p_2})^p \\
&\quad + m_{2,p} \int_0^1 (\|C(2, s, x, y)\|_{p_1} \|B^x(s) - B^y(s)\|_{p_2})^p ds \\
&\leq m_{1,p} c_0(x, y, p_1)^{p/p_1} m_{p_2}^{p/p_2} |x - y|^p \\
&\quad + m_{2,p} c_2(x, y, p_1)^{p/p_1} m_{p_2}^{p/p_2} |x - y|^p.
\end{aligned}$$

Next, fixing

$$M(p, p_1, p_2) = \sup_{x \neq y} \left( m_{1,p} c_0(x, y, p_1)^{p/p_1} m_{p_2}^{p/p_2} + m_{2,p} c_2(x, y, p_1)^{p/p_1} m_{p_2}^{p/p_2} \right),$$

we have

$$\|f(x) - f(y)\| \leq M(p, p_1, p_2) |x - y|^p \quad (6.57)$$

Now for  $p$  large enough, we can use Theorem 1.4.1 in [7], to deduce that there exists  $\tilde{f}$ , which is the continuous modification of  $f$ . Abusing the notation, we shall write  $f$  for  $\tilde{f}$ . Also, using the same result, we can infer that the modulus of continuity of  $f$  satisfies

$$P(\Xi_f(\eta) > \epsilon) = o(\eta^{\dim(M)}) \text{ as } \eta \downarrow 0,$$

for any  $\epsilon > 0$ .

Note that we needed supremum of  $c_2(x, y)$  to be bounded to prove the continuity of  $f$ , and to control its modulus of continuity. we can further infer that the conditions stated in (A1) – (A4) suffice to obtain similar results for the modulus of continuity of  $\nabla f$  and  $\nabla^2 f$ .  $\square$

Recall from [2] that  $\mathcal{L}_0$ , also known as the Euler–Poincare characteristic, of the excursion set  $A_u(f; M)$ , can be expressed as<sup>16</sup>

$$\begin{aligned}\mathcal{L}_0(A_u(f; M)) &= \sum_{k=0}^m (-1)^k \#\{x \in M : f(x) \geq u, \nabla f(x) = 0, \text{index}(\nabla^2 f) = k\} \\ &= \sum_{k=0}^m (-1)^k \mu_k.\end{aligned}$$

Next, we shall state the most important result for proving a GKF, called the *expectation metatheorem*, which can be stated as follows, and the proof of which can be found in Theorem 11.2.1 of [2].

**Lemma 6.4** *Let  $M$  and  $B$  be subsets of  $\mathbb{R}^K$  and  $\mathbb{R}^{K'}$ , respectively, such that,  $\partial M$  and  $\partial B$ , boundaries of the sets  $M$  and  $B$  have finite  $(K-1)$  and  $(K'-1)$  dimensional Hausdorff measure, respectively. Also, let  $G = (G^1, \dots, G^K)$  and  $H = (H^1, \dots, H^{K'})$  be two  $N$  parameter random fields satisfying the following conditions*

- (i) **Continuity:**  $G, \nabla G, H$  are almost surely continuous, and have finite variances.
- (ii) **Moduli of continuity:** The moduli of continuity  $\Xi$  with respect to the Euclidean norm of each component  $G, \nabla G, H$  satisfy

$$P(\Xi(\eta) > \epsilon) = o(\eta^K) \quad \text{as } \eta \downarrow 0$$

for any  $\epsilon > 0$ .

- (iii) **Marginal densities:** For all  $x \in M$ , the marginal densities  $p_{G(x)}(y)$ ,  $p_{\nabla G(x)}(y)$  and  $p_{H(x)}(y)$ , of  $G(x)$ ,  $\nabla G(x)$ , and  $H(x)$ , respectively, are continuous, for each  $x \in M$ .
- (iv) **Conditional densities:** Let us assume that the joint density of  $(G, \nabla G, H)$  exists, and that it is positive on the set  $\mathcal{O}$ . Also, assume
  - existence of the conditional density  $p_{G(x)|\nabla G(x), H(x)}(u|v, w)$ , and continuity in  $u$ .
  - continuity in  $v$  and  $w$  of the conditional density  $p_{G(x)|\nabla G(x), H(x)}(u|v, w)$  in the set  $\mathcal{O}$ .
  - continuity in  $v$  and  $w$  of the conditional density  $p_{\det \nabla G(x)|G(x)}(v|u)$  in the set  $\mathcal{O}$ .
  - continuity in  $v$  and  $w$  of the conditional density  $p_{H(x)|G(x)}(w|u)$  in the set  $\mathcal{O}$ .

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<sup>16</sup>This particular disintegration of the Euler–Poincare characteristic is based on the assumption that the underlying random field is a Morse function. For a detailed account of this and more, we refer the reader to Chapters 8 and 9 of [2].

(v) **Moment condition:**

$$\sup_{x \in M} \max_{1 \leq i, j \leq K} E|G_j^i(x)| < \infty,$$

$$\text{where } G_j^i = \frac{\partial G^i}{\partial x_j}.$$

Then, writing  $N_u$  for the upcrossings defined below

$$N_u \equiv N_u(G, H : M, B) \triangleq \{x \in M : G(x) = u \in \mathbb{R}^K \text{ and } H(x) \in B \subset \mathbb{R}^{K'}\},$$

we shall have

$$E(N_u) = \int_M E(|\det \nabla G(x)| 1_B(H(x)) |G(x) = u) p_{G(x)}(u) dx.$$

Now replacing  $G$  and  $H$  by  $\nabla f$  and  $(\nabla^2 f, f)$  respectively, and  $B$  by  $D_k \times [u, \infty)$ , where  $D_k$  is the space of  $m \times m$  matrices with index  $k$ , we can obtain a formula for the expected value of  $\mu_k$  as defined above. However, in order to use the above theorem for our purpose, we first need to check the regularity conditions involved in the above theorem. Conditions (i) and (ii) are consequences of Theorem 6.3, whereas condition (iii) is a simple consequence of the Sobolev smoothness of the random fields  $f$  (cf. [8] or [9]), and (v) is trivially satisfied due to the smoothness of the field. Finally for conditions (iv), we shall state the following result from [10].

**Lemma 6.5** *Let  $(F^1, \dots, F^a)$  and  $(G^1, \dots, G^b)$  be elements from Sobolev space  $D_1^2$  of the Wiener space, such that the Malliavin matrices of the augmented vector  $(F^1, \dots, F^a, G^1, \dots, G^b)$  and  $(G^1, \dots, G^b)$  have positive determinant. Then, there exists a conditional density for the law of  $(F^1, \dots, F^a)$  given the  $\sigma$ -field generated by  $(G^1, \dots, G^b)$ .*

Thus, using the above lemmas together with Theorem 6.3, we can write

$$\begin{aligned} & E(\mathcal{L}_0(A_u(f; M))) \\ &= \sum_{k=0}^N (-1)^k \int_M E(|\det \nabla^2 f(x)| 1_{D_k}(\nabla^2 f(x)) 1_{[u, \infty)}(f(x)) |\nabla f(x) = 0) p_{\nabla f(x)}(0) dx \\ &= \int_M E(\det(-\nabla^2 f(x)) 1_{[u, \infty)}(f(x)) |\nabla f(x) = 0) p_{\nabla f(x)}(0) dx \end{aligned} \tag{6.58}$$

Next, in order to construct an approximating sequence to the LHS of (6.56), and appeal to the earlier results in [2], we shall use a cylindrical approximation of  $f(x)$ . Writing  $\pi_n$  as a sequence of partitions of the set  $(0, 1]$ , given by  $\{(i/n, (i+1)/n)\}_{i=0}^{n-1}$ , with the size of the partition  $\pi_n$  going to zero as  $n \rightarrow \infty$ , let us define

$$f_n(x) = \sum_{i=0}^{n-1} V(B^x(i/n))(B^x((i+1)/n) - B^x(i/n)).$$

Standard results from stochastic analysis ensure the convergence of  $f_n(x)$  to  $f(x)$ . Moreover, note that  $(B^x((i+1)/n) - B^x(i/n))_{i=0}^{n-1}$  forms an i.i.d.  $0 \leq i \leq (n-1)$ . Therefore, we can write

$$f_n(x) = F_n(y_1^{(n)}(x), \dots, y_n^{(n)}(x)),$$

where  $y_{i+1}^{(n)}(x)$  are i.i.d. with the same distribution as  $\sqrt{n}(B^x((i+1)/n) - B^x(i/n))$ , and  $F_n$  is a real valued function, given by

$$F_n(y_1, \dots, y_n) = n^{-1/2} \sum_{i=1}^n V\left(n^{-1/2} \sum_{j=1}^{i-1} y_j\right) y_i.$$

Under the conditions imposed on  $f$  for the expectation metatheorem to be true,  $f_n$  also becomes a valid candidate to apply the metatheorem, thereby giving us

$$\begin{aligned} & E(\mathcal{L}_0(A_u(f_n; M))) \\ &= \sum_{k=0}^N \int_M E(\det(-\nabla^2 f_n(x)) 1_{[u, \infty)}(f(x)) |\nabla f_n(x) = 0) p_{\nabla f_n(x)}(0) dx. \end{aligned} \quad (6.59)$$

Using Theorem 15.9.5 of [2] for the random field  $f_n$ , we shall have

$$E(\mathcal{L}_0(A_u(f_n; M))) = \sum_{j=0}^m (2\pi)^{-j/2} \mathcal{L}_j(M) \mathcal{M}_j^\mu(F_n^{-1}[u, \infty)), \quad (6.60)$$

where  $F_n^{-1}[u, \infty)$  is a subset of  $\mathbb{R}^n$ .

**Theorem 6.6** *Let  $\{G_n\}_{n \geq 1}$  be a sequence of real valued Wiener functionals, such that  $G_n$  belongs to the  $n$ -th Wiener chaos<sup>17</sup>, and  $G_n \rightarrow G$  in  $D_3^{\infty-}$ , for some  $G \in D_3^{\infty-}$ . Also, let that each  $G_n$  and  $G$  satisfy all the assumptions of Theorem 4.3 then  $\mathcal{M}_j^k(G_n^{-1}[u, \infty)) \rightarrow \mathcal{M}_j^\mu(G^{-1}[u, \infty))$ , as  $n \rightarrow \infty$ .*

**Proof:** Let us start with the definition of the GMFs in Theorem 4.3, and we can conclude that it suffices to prove the following

$$\begin{aligned} & \int_{G^{-1}(u)} \det_2(I_H + rD\eta_n) \exp\left(-r\delta(\eta_n) - \frac{1}{2}r^2\right) da^{G_n^{-1}(u)} \\ & \rightarrow \int_{G^{-1}(u)} \det_2(I_H + rD\eta) \exp\left(-r\delta(\eta) - \frac{1}{2}r^2\right) da^{G^{-1}(u)}, \end{aligned}$$

where  $\eta = DG/\|DG\|_H$  and  $\eta_n = DG_n/\|DG_n\|$ . Together with the fact that the densities  $p_{G_n}$  converge to  $p_G$ , the density of  $G$ , the above can further to simplified to proving

$$\begin{aligned} & E^{G_n=u}([\det(\sigma_{G_n})]^{1/2} \det_2(I_H + rD\eta_n) \exp(-r\delta(\eta_n) - \frac{1}{2}r^2)) \\ & \rightarrow E^{G=u}([\det(\sigma_G)]^{1/2} \det_2(I_H + rD\eta) \exp(-r\delta(\eta) - \frac{1}{2}r^2)). \end{aligned}$$

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<sup>17</sup>We have used the definition of the  $n$ -th chaos as it appears on p.17 of [8]

Now using the relationship between the conditional expectation and positive generalized Wiener functionals from Section 4, in particular recalling Theorem 4.1, and writing  $A_n = ([\det(\sigma_{G_n})]^{1/2} \det_2(I_H + rD\eta_n) \exp(-r\delta(\eta_n) - \frac{1}{2}r^2))$ , and similarly defining  $A$ , let us consider

$$\begin{aligned}
& |E^{G_n=u} A_n - E^{G=u} A| \\
&= |E(A_n \delta_u \circ G_n) - E(A \delta_u \circ G)| \\
&\leq |E(A_n \delta_u(G_n)) - E(A_n \delta_u(G))| + |E(A_n \delta_u(G)) - E(A \delta_u(G))| \\
&\leq \|A_n\|_{D_\alpha^{p/(p-1)}} \|\delta_u(G_n) - \delta_u(G)\|_{D_{-\alpha}^p} + \|A_n - A\|_{D_\alpha^{p/(p-1)}} \|\delta_u(G)\|_{D_{-\alpha}^p},
\end{aligned} \tag{6.61}$$

where we recall Theorem 4.1 for definitions of  $p$  and  $\alpha$ . We also note that, since  $G_n$  and  $G$  are elements of  $D_3^{\infty-}$ , and that they are non-degenerate, existence of such  $p$  and  $\alpha$  is ensured. Moreover, since  $G_n \rightarrow G$  in  $D_3^{\infty-}$ , it's easy to see that  $\sup_n \|A_n\|_{D_\alpha^{p/(p-1)}} < \infty$ , and  $\|A_n - A\|_{D_\alpha^{p/(p-1)}} \rightarrow 0$ . Also  $\|\delta_u(G_n) - \delta_u(G)\|_{D_{-\alpha}^p} \rightarrow 0$ , which proves the result.  $\square$

**Proof of Theorem 6.1 continued:** Extending  $F_n$  from  $\mathbb{R}^n$  to  $\mathbb{R}^\infty$ , or equivalently to  $X$ , by suppressing all the indices after the first  $n$ , i.e., considering  $F_n$  as cylindrical Wiener functionals. Then, by using the invariance property of GMFs, the  $\mathcal{M}_j^\mu$ 's of the extended  $F_n$  remain the same as that of  $F_n$  when restricted to  $\mathbb{R}^n$ . Together with this, using the fact that  $F_n$  converges to  $F$  in  $D_3^{\infty-}$ , we clearly have

$$\lim_{n \rightarrow \infty} \mathcal{M}_j^\mu(F_n^{-1}[u, \infty)) = \mathcal{M}_j^\mu(F^{-1}[u, \infty)), \tag{6.62}$$

Therefore, using (6.60) and (6.62), we shall have

$$\lim_{n \rightarrow \infty} E(\mathcal{L}_0(A_u(f_n; M))) = \sum_{j=0}^N c_j \mathcal{L}_j(M) \mathcal{M}_j^\mu(F^{-1}[u, \infty)). \tag{6.63}$$

Now, it suffices to prove that the right hand side of (6.59) converges to the right hand side of (6.58). Clearly,

$$\lim_{n \rightarrow \infty} p_{\nabla f_n}(y) = p_{\nabla f}(y), \tag{6.64}$$

which follows from the fact that  $\nabla f_n$  converges to  $\nabla f$  in a much stronger sense as is clear from the assumption  $f_n \rightarrow f$  in  $D_{3+\delta}^\infty$ .

Next, we need to prove

$$\begin{aligned}
& |E(|\det \nabla^2 f_n(x)| 1_{D_k}(\nabla^2 f_n(x)) 1_{[u, \infty)}(f_n(x)) | \nabla f_n(x) = 0) \\
& - E(|\det \nabla^2 f(x)| 1_{D_k}(\nabla^2 f(x)) 1_{[u, \infty)}(f(x)) | \nabla f(x) = 0)| \\
& \rightarrow 0,
\end{aligned} \tag{6.65}$$

which is similar to the proof of Theorem 6.6. Using precisely the same techniques, and writing  $B_n(x) = (|\det \nabla^2 f_n(x)| 1_{D_k}(\nabla^2 f_n(x)) 1_{[u, \infty)}(f_n(x)))$ , and



defining  $B(x)$  in a similar fashion, we have

$$\begin{aligned}
& |E^{\nabla f_n(x)=0} B_n(x) - E^{\nabla f(x)=0} B(x)| \\
& \leq |E^{\nabla f_n(x)=0} B_n(x) - E^{\nabla f(x)=0} B_n(x)| + |E^{\nabla f(x)=0} B_n(x) - E^{\nabla f(x)=0} B(x)| \\
& = |E(B_n(x)\delta_0(\nabla f_n(x))) - E(B_n(x)\delta_0(\nabla f(x)))| \\
& \quad + |E(B_n(x)\delta_0(\nabla f(x))) - E(B(x)\delta_0(\nabla f(x)))| \\
& \leq \|B_n(x)\|_{L^{p/(p-1)}} \|\delta_0(\nabla f_n(x)) - \delta_0(\nabla f(x))\|_{L^p} \\
& \quad + \|B_n(x) - B(x)\|_{L^{p/(p-1)}} \|\delta_0(\nabla f(x))\|_{L^p},
\end{aligned} \tag{6.66}$$

which, under the assumptions of  $f_n(x) \rightarrow f(x)$  in  $D_3^{\infty-}$ , and non-degeneracy of  $\nabla f(x)$ , converges to zero as  $n \rightarrow \infty$ .

This proves that the integrand of (6.59) converges to that of (6.58) for each  $x \in M$ .

Finally, in order to prove that the integral involved in the equation (6.59) converges to the integral in (6.58), note that the random fields  $f_n$  and  $f$  defined on the manifold  $M$  are chosen to be sufficiently smooth so that we can use uniform integrability argument to conclude that the right hand side of (6.59) converges to the right hand side of (6.58). Therefore, we shall have

$$\begin{aligned}
E(\mathcal{L}_0(A_u(f; M))) &= \lim_{n \rightarrow \infty} E(\mathcal{L}_0(A_u(f_n; M))) \\
&= \lim_{n \rightarrow \infty} \sum_{j=0}^m c_j \mathcal{L}_j(M) \mathcal{M}_j^\mu(F_n^{-1}[u, \infty)) \\
&= \sum_{j=0}^m c_j \mathcal{L}_j(M) \mathcal{M}_j^\mu(F^{-1}[u, \infty)),
\end{aligned}$$

where in going from first line to the second, we have used the finite dimensional results set forth in [2], and in going from second to the third line we have used Theorem 6.6.  $\square$

**Remark 6.7** *Using a Gaussian Crofton formula as it appears in Chapter 13 of [2], we can extend Theorem 6.1 to  $\mathcal{L}_i(A_u(f; M))$  as*

$$E(\mathcal{L}_i(A_u(f; M))) = \sum_{j=0}^{m-i} \binom{i+j}{j} \frac{\omega_{i+j}}{\omega_i \omega_j} (2\pi)^{-j/2} \mathcal{L}_{i+j}(M) \mathcal{M}_j^\mu(F^{-1}[u, \infty)),$$

where  $\omega_l$  is the volume of a unit ball in  $\mathbb{R}^l$ .

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